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Efficient simulations with Quasi Monte-Carlo Sequences

According to the *Beroepsprofiel Actuaris AG* the first core task of an actuary is actually a combination of tasks, namely to perform valuation and to determine capital [1]. In both cases Monte-Carlo simulations play an important role [2]. When performing valuation of (embedded) options, often risk-neutral simulations are used. The expected value of discounted pay-offs then determines the market-consistent value. When determining the capital with an Internal Model, often real-world simulations are used. The 99.5% worst-case scenario over a one-year horizon then determines the capital. An important disadvantage of Monte-Carlo is that it converges slowly. There are distinct types of approaches to tackle this problem. The first approach is to use the ever increasing computing power in the cloud to generate as many scenarios as possible. The second approach is to look for ways to improve convergence speed. In this article we discuss the second approach, which is an especially suitable approach, when resources are scarce.

(QUASI) MONTE-CARLO SIMULATION

In Monte-Carlo simulations we use random sampling to estimate the quantity of our interest. The name of the method refers to the grand casino at Monte Carlo, where random experiments called gambling are performed. If we want to know the mean of a random variable, and if we are able to simulate that random variable, then we can use the law of large numbers to approximate the expected value by the sample mean over many simulations. The advantage of this procedure is that it is very generally applicable. The disadvantage is that the accuracy of the sample mean over independent contributions only improves with $\sqrt{n}$ with n the number of simulations. There have been many well-known variance reduction techniques developed that improve accuracy. However, the error reduction typically remains proportional to $1/\sqrt{n}$ [2].

Theoretically, if we determine the expected value of a stochastic variable, then we need to integrate the outcomes of that variable with respect to the probability density. Numerically, we perform integration by performing a summation over grid points, where we prefer to choose a grid that minimizes error. A random sequence of points is not optimal for performing integration, because it can leave large 'holes', resulting in large errors. This is shown in Figure 1, where the left plot shows the points that are used in Monte Carlo simulation and the right plot shows the points in a so-called Quasi-Monte Carlo sequence based on the method of Sobol [3].

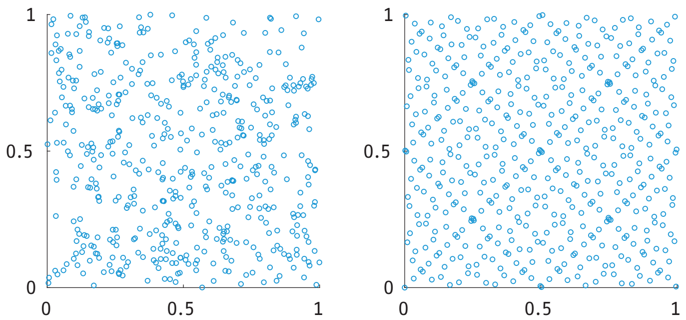

Figure 1: example of a random number sequence (left plot) and a Sobol sequence (right plot).

Quasi-Monte Carlo sequences do not try to mimic randomness, but they are rather designed to make the 'holes' between grid points as small as possible. Therefore, they are also called low-discrepancy sequences. In one dimension, such a sequence would start with 1/2, after which 1/4 and 3/4 would be added, after which 1/8, 5/8, 3/8 and 7/8 would added, etc. This is called a Van der Corput-sequence, which gives a simple example of how to fill holes in one dimension. In higher dimensions, the main tasks become to find appropriate directions and to fill these directions in the appropriate order. Various methods exist, but one of the most popular methods is by Sobol [3], due to its good performance in practice [2]. The accuracy of the method scales with $\mathcal{O}\left(\frac{\log(n)^d}{n}\right)$ [2], which can significantly improve the accuracy compared to $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ in low dimensions d. Furthermore, it is possible to randomize Sobol sequences by randomly permuting simulated subintervals [2]. This leads to so-called scrambled Sobol sequences, which combine favorable properties of random and low-discrepancy sequences. For example, it allows us to statistically study simulation errors.

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APPLICATION TO RISK-NEUTRAL VALUATION

As a first application, we consider risk-neutral valuation. To this end, we start from a financial contract with a pay-off function X_T at maturity T. The canonical example is a call option on a stock price S_T with strike K for which the pay-off is given by: $X_T = \max(S_T - K, 0)$. In a risk neutral scenario all simulated asset prices earn the risk free rate. After discounting with factor D_T, the asset prices become driftless processes, called martingales. As a result, the value V_T of self-financing investment strategies become martingales after discounting. This leads to the following general formula for the initial value V_0 of a financial contract in terms of a self-financing replicating portfolio: $V_0 = \mathbb{E}_{\mathbb{Q}}[D_T V_T] = \mathbb{E}_{\mathbb{Q}}[D_T X_T]$. The first step in this equation uses the self-financing (martingale) property and the second step uses the replicating property: $V_T = X_T$. The risk neutral valuation formula can numerically be implemented using standard Monte-Carlo techniques by simulating discounted pay-offs and taking the sample mean.

We study the convergence of the call option price using the Black-Scholes-Merton model. The advantage is that this price is analytically known, so that we can compare the accuracy of the numerical techniques to the exact price. We use a risk-free rate of r = 2%, a volatility parameter of σ = 25%, an initial stock price of S_0 = 100, a strike of K = 105 and a maturity of T = 3 years. The results for the accuracy are shown in Figure 2. For each number of scenarios, we calculate the option value 50 times to determine the Root Mean Squared Error relative to the analytical value. We see that the Sobol sequence has much lower error (more than an order of magnitude improvement) and converges faster than the standard Monte-Carlo sequence. Sobol sequences are therefore often considered best practice in quantitative finance applications [2]. The observed accuracy is also seen to match the expected theoretical behavior.

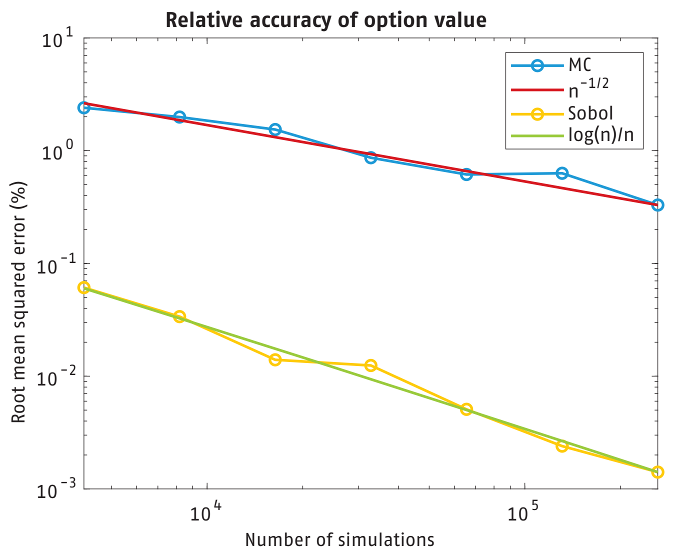

Figure 2: Accuracy of Monte-Carlo and (scrambled) Sobol sequences in valuation of a call option.

APPLICATION TO DETERMINE CAPITAL

As a second application, we consider the determination of capital. For the Solvency Capital Requirement (SCR), we require a model for the risks that affect the own funds over a one-year horizon. This no longer involves the study of self-financing portfolios with martingales, so we turn from risk-neutral to real-world scenarios that include drift. The SCR is defined by the Value-at-Risk (VaR) at a confidence level of 99.5%. This means that the probability of the loss L exceeding l_{SCR} is given by 0.5% ($\mathbb{P}[L > l_{SCR}]$ = 0.5%). To obtain the distribution function $\mathbb{P}$, we have to perform an integral over the probability density. Therefore, Sobol sequences can again be used to improve accuracy.

We study a hypothetical insurer which has a portfolio driven by five risk factors with equal exposure. The sum over these risk factors determines the total loss. The factors satisfy a multivariate t-distribution with 4 degrees-of-freedom and a correlation parameter of 0.5 between all pairs. The t-distribution leads to heavy tails and tail dependence, so that it is suitable for studying capital requirements. Another advantage of this example is that the SCR can be calculated analytically. We simulate the distribution with random sequences and Sobol sequences. The VaR is obtained by sorting the scenarios and taking the 99.5% largest value of the loss. The results for the accuracy are shown in Figure 3. For each number of scenarios, we have calculated the SCR 50 times to determine the Root Mean Squared Error relative to the analytical value of the SCR. We again see that the (scrambled) Sobol sequences have higher accuracy than the standard Monte-Carlo approach. In both cases, the observed accuracy matches the expected theoretical behavior.

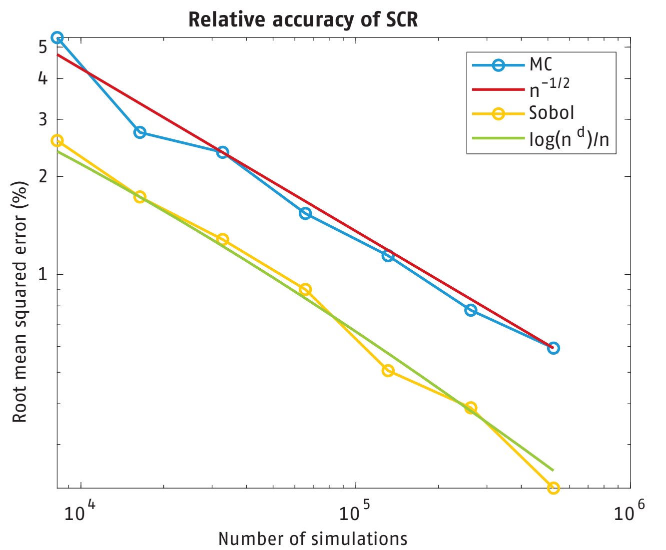

Figure 3: Accuracy of Monte-Carlo and (scrambled) Sobol sequences in determining the SCR.

CONCLUSION

We have studied the application of Sobol sequences in performing valuation and determining capital, which is the first core task of the actuary. It is possible to achieve higher accuracy with less resources by applying Sobol sequences in Monte-Carlo calculations. Another advantage is that the implementation only requires applying a different set of (quasi-)random numbers, which are readily available in most used programming languages. When the required accuracy is high, but resources are scarce, these techniques can be of particular relevance for insurance companies. ■

Referenties

[1] Koninklijk Actuarieel Genootschap (2016), *Beroepsprofiel Actuaris AG*
[2] P. Glasserman (2004), *Monte Carlo methods in financial engineering*
[3] I.M. Sobol (1967), *Distribution of points in a cube and approximate evaluation of integrals*