Welfare Losses of Near-Optimal Investment Strategies in Incomplete Markets

by

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Abstract

This thesis addresses the portfolio optimization problem of a CRRA investor who derives utility from terminal wealth, within the incomplete financial market models of Koijen et al. (2010), Heston (1993), and Commissie Parameters (2022), using a combination of the Martingale method and convex duality theory. Based on the optimal investment strategies in the models of Koijen et al. (2010) and Heston (1993), we analyze the fundamental welfare risks inherent in the risk preference research, which is mandatory under the new Dutch pension law. We find that a myopic investor in the model of Koijen et al. (2010) faces significant welfare losses, whereas a myopic investor in the model of Heston (1993) faces minimal welfare losses. When investment is based on an incorrect risk aversion parameter, the resulting welfare losses are roughly comparable across the two models. Welfare losses become significant as the gap between the true and estimated value of the risk aversion parameter widens. We argue that the portfolio optimization problem in the model of Commissie Parameters (2022) cannot be solved analytically on the basis of currently available techniques.

Preface

Presenting this thesis marks the conclusion of my time as a student. The past six years have been enjoyable, challenging, and rewarding, sometimes all at once. Throughout the past years, especially during the writing of this thesis, I have gained valuable academic and personal insights, for which I am grateful to several people.

The completion of this thesis would not have been possible without the guidance of my supervisors, Prof. Dr. Bas Werker and Dr. Thijs Kamma. I am thankful for the opportunity to work together and to learn from your deep expertise. I am particularly grateful to Bas for his reflective feedback, continuous flow of ideas, and his guidance during the writing process. This showed me how to approach a thesis in an academic and professional manner. I would like to especially thank Thijs for all the nuanced answers he gave to my questions and his ability to place my work in a broader perspective. Every time I believed I had reached a solid conclusion, Thijs would offer a new perspective, leading to interesting conversations.

I would like to thank my parents, other family, and friends for their support during the last six years. It provided significant support during difficult periods. I am truly thankful to my friends Casper, Tim, and Wim for their support. I had the privilege of working closely with Casper on numerous assignments during our Bachelor's studies. Tim and Wim greatly supported me by reviewing my thesis for coherence and textual mistakes, this way serving as a personalized large language model, more precise than any tool could ever be. Above all, it was the personal relationship that I valued the most, far beyond academic feedback.

Lastly, I would like to thank Ronald from AZL for facilitating the thesis process and expressing his faith and support throughout the process.

Transparency of AI Use

Regarding the use of AI, I have used ChatGPT in Section 2.3, Section 2.4, Section 3.3, Section 3.4, and Section 4.3 for the purpose of asking coding questions about numerical implementations in MATLAB and for assistance with creating LaTeX-tables. Furthermore, I have used ChatGPT to review existing sentences on grammar and spelling errors.

Contents

Chapter 1

Introduction

In 2019, the Dutch government, social partners, and Social-Economic Counsil (SER) reached an agreement on the reform of the Dutch pension system. Under the new Dutch Pension Law (WTP), pension funds are required to assess the risk preferences of their participants using a risk preference research (Ministerie van Sociale Zaken en Werkgelegenheid, 2022b). The goal of a risk preference research is to determine both a participant's risk aversion level and risk-bearing capacity so that an adequate life cycle can be formed. Hence, the risk preference research enables a better alignment of the pension system with participants' individual preferences under WTP (Ministerie van Sociale Zaken en Werkgelegenheid, 2022b). The pension industry currently uses the theoretical framework of Merton (1969) as the foundation of the risk preference research. This framework relies heavily on two theoretical assumptions: (i) the use of Constant Relative Risk Aversion (CRRA) utility and (ii) the use of a Black-Scholes financial market, i.e. the market consists of one traded asset whose price follows a Geometric Brownian motion. The optimal life cycle derived from this framework is partly determined by the participant's risk aversion and risk-bearing capacity, which are measured by the risk preference research. One of the goals of the risk preference research is to estimate the true risk aversion parameter of the participants as accurately as possible. Since the estimated risk aversion directly influences the life cycle, a wrong estimate will result in investment strategies that are either too aggressive or too conservative for the participant. Hence, wrong estimates of a participant's risk aversion parameter will lead to welfare losses.

In addition to the risk parameters, the optimal life cycle in the framework of Merton (1969) is influenced by the interest rate, and the mean and volatility of stock returns. All of these economic factors are assumed to be deterministic such that the only randomness originates from the traded asset. In practice, the pension industry uses the quarterly published scenario sets by the Dutch Central Bank (DNB) to forecast financial market developments. The scenario sets are generated with the model of Commissie Parameters (2022), which builds upon the model of Koijen et al. (2010). The financial market model of Koijen et al. (2010) accommodates uncertainty about the interest rate, expected inflation, customer price index, and stock returns. Commissie Parameters (2022) extends this framework by integrating stochastic volatility. This implies an inherent mismatch between the financial market model underlying the life cycle formula and the financial market model from which the necessary economic quantities are derived. Consequently, interest rate risk, inflation risk, and volatility risk are not taken into account in the current life cycle, resulting in welfare losses.

Life cycles under WTP are thus sensitive to inaccurate estimates of the risk aversion parameter and/or the omission of certain financial risks. Therefore, the purpose of this thesis is to analyze the welfare effects of using an inaccurate risk aversion parameter and neglecting interest rate risk, inflation risk, and volatility risk. To give a structured overview of the effects of each of the risks, we present three different models: (i) the stochastic interest and inflation framework by Koijen et al. (2010), (ii) the stochastic volatility framework by Heston (1993), and (iii) the model by Commissie Parameters (2022) that combines stochastic interest rates, inflation, and volatility. For the models of Koijen et al. (2010) and Heston (1993) we will present the financial market, solve the optimal investment problem of a CRRA investor receiving utility from terminal wealth, numerically analyze the optimal investment strategy, and present the welfare losses of ignoring the underlying risks or using an inaccurate risk parameter. This method allows for (i) quantification of the welfare effects following from the current mismatch between financial market models underlying the risk preference research, (ii) insight in the welfare effects of inaccurately estimating participants' risk aversion parameters, and (iii) a comparison of the magnitude of both of the welfare effects. For the model of Commissie Parameters (2022) we will present the financial market and argue that the optimal investment problem of a CRRA investor cannot be solved in this market. Therefore, we will present an estimation of the optimal strategy that could form the basis for future research on optimal investment in the model of Commissie Parameters (2022).

This thesis contributes to the existing literature in several ways. First of all, to the best of our knowledge, no prior research has examined optimal investment within the model of Commissie Parameters (2022). Therefore, the findings on the optimal investment problem of a CRRA investor in this framework are novel. Furthermore, we extend the literature by solving the optimal investment problem in the incomplete market model of Koijen et al. (2010) and Heston (1993) with a combination of the Martingale method and convex duality theory, which is introduced in the literature by Karatzas et al. (1991), Cvitanic and Karatzas (1992), Xu and Shreve (1992), and Kamma (2023). Although optimal investment has been studied in the model of Heston (1993), and while the model of Koijen et al. (2010) has not been directly addressed, related models have been explored, none of these studies solve the problem in an incomplete market using duality theory. Therefore, we argue that the approach taken in this thesis represents a methodological refinement to address the optimal investment problem of a CRRA investor in such incomplete markets. Lastly, we contribute to the literature by providing an extensive welfare analysis in the context of the risk preference research in the models of Koijen et al. (2010) and Heston (1993).

Each of the three financial market models will be presented in a separate chapter: in Chapter 2 we will present the findings in the model of Koijen et al. (2010), in Chapter 3 we will present the results in the model of Heston (1993), and in Chapter 4 we will provide the results in the model of Commissie Parameters (2022). In each of these chapters we will discuss the most relevant findings in the literature related to the model of interest. The rest of this chapter is used to present some overarching concepts relevant throughout this thesis. In Section 1.1 we will focus on the foundations of Merton's initial work on (continuous time) portfolio optimization. Merton (1969, 1971, 1973) operates in complete financial markets, whereas the financial markets we consider are incomplete. Hence, Section 1.2 is used to discuss the role of market incompleteness in this thesis, specifically in relation to the Martingale method and duality theory. Finally, we are interested in the welfare effects of near-optimal investment strategies. Therefore, we will elaborate on the framework used to quantify welfare losses in Section 1.3.

1.1 Foundations of portfolio optimization

Portfolio optimization leads to portfolio weights that maximize the expected utility an agent receives from its portfolio in a mathematically stated financial market, for a given risk aversion level (Bruce and Greene, 2013). The work of Merton (1969, 1971, 1973) forms the basis for research on such optimization problems in continuous time. Until this sequence of papers most of the research on portfolio optimization was focused on one-period discrete models (Merton, 1969). In his first paper, Merton extends the one-period framework to a multi-period continuous time model in which the agent seeks to maximize expected utility over a finite horizon, deriving utility from both consumption and terminal wealth. In this thesis we will solely focus on the situation where the agent receives utility from terminal wealth. We make this choice for mathematical convenience; intermediate consumption introduces additional terms in the optimization problem, making the problem less tractable. Similar to the work of Merton (1969, 1971, 1973), we expect that all results in this thesis can be extended in a straightforward manner to the case where the agent (also) receives utility from intermediate consumption. In the context of a pension scheme where the accumulation phase is separated from the retirement phase, the terminal wealth problem can be understood as the situation in which the agent receives a lump-sum payment at retirement.

The main theoretical concepts underlying Merton's continuous time approach are (i) the use of a Black-Scholes financial market and (ii) the use of the class of power utility functions to describe preferences. The Black-Scholes financial market consists of a stock and a risk-free asset. For the traded asset it is assumed that the price follows a Geometric Brownian Motion so that the distribution of stock prices is log-normal. The risk-free asset pays the constant interest rate (Black and Scholes, 1973). The class of power utility functions is described as follows[1]:

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \ln x & \text{else} \end{cases} \tag{1.1}$$

where x is the level of wealth or consumption and γ the parameter describing the level of risk aversion of the agent. The alternative definition of the utility function for $\gamma = 1$ introduces the need to formally define two optima. Although we expect that the optimum for $\gamma = 1$ is the same as the optimum that results from using $\gamma = 1$ in the optimum for $\gamma \neq 1$[2], this should be mathematically proved. This leads to more mathematical derivations without new fundamental insights and thus we choose to only consider the cases where $\gamma \neq 1$ in this thesis. The popularity of the class of power utility functions stems from its constant relative risk (CRRA) property and its analytical tractability (Joseph et al., 2020). The CRRA property implies that the coefficient of relative risk is independent of the agents wealth level. Intuitively this means that agents perceive changes in wealth independent of their level of wealth and thus risk perception is independent of the level of wealth (Arrow, 1965).

Combining the properties of the Black-Scholes market and the power utility function allowed Merton (1969) to solve the continuous time optimization problem. Considering an agent who receives utility from terminal wealth only in a financial market with one risky asset, the optimal fraction of wealth invested in this asset equals:

$$\theta = \frac{\mu - r}{\sigma^2 \gamma} \tag{1.2}$$

where μ is the expected arithmetic return of the risky asset, r the risk-free rate, and σ the constant volatility of the stock. We thus see that the fraction of wealth invested in the risky asset is independent of the time horizon. Note however that in the context of a pension scheme, total wealth includes both past contribution payments (financial wealth) and human capital (contribution payments to be made in the future). Merton's optimal strategy prescribes that the optimal fraction of total wealth invested in the risky asset does not change over time. However, as the proportion of financial wealth to total wealth changes over time, an agent's risk-bearing capacity will change, and thus the

[1]Note that for $\gamma = 1$ the power utility function is defined as the limit of γ approaching 1, i.e. $U(x) = \lim_{\gamma \to 1} \frac{x^{1-\gamma}-1}{1-\gamma} = \ln x$.

[2]In other words, we expect that taking the limit of γ approaching 1 and then finding the optimum is mathematically equivalent to finding the optimum and then taking the limit of γ approaching 1.

fraction of financial wealth invested in the risky asset will typically change over time (Joseph et al., 2020).

The framework used by Merton (1969) and Black and Scholes (1973) is characterized by the fact that the parameters describing the asset dynamics remain constant over time. While this enhances the model's tractability, the effect of more realistic assumptions remains to be answered. In Merton (1971, 1973) the original framework is generalized to explore the effects of a broader class of financial assumptions and utility functions. In the light of this thesis, an important finding is the so-called mutual fund separation theorem. This theorem states that if an agent can invest in a risk-free asset and risky assets whose randomness is realized by Geometric Brownian motions with constant parameters, the optimal portfolio can be formed by holding a position in the risk-free asset and a mutual fund (Merton, 1971). In this context, a mutual fund is an asset that pools together all risky assets. We thus see that optimality can be achieved by holding less assets than the total number of available assets (Merton, 1971). An important consequence of the mutual fund separation theorem is that the relative allocation to the assets in the mutual fund is the same for each agent, regardless of its preferences and wealth (Merton, 1971). This is a consequence of the fact that the mutual fund is mean-variance efficient. Therefore, all agents hold the same mutual fund, only differing in which share of their total wealth they allocate to it, based on different risk aversion parameters. In the rest of Merton's 1971 paper, different generalizations of the Merton (1969) framework are studied, for example the effect of stochastic labor income, uncertainty in life expectancy, and the use of generalized utility functions. In general, portfolio strategies will change significantly when extending the model (Merton, 1973). We refer to Merton (1971) for further information on the generalized models.

In Merton (1973) the continuous time portfolio problem is put into a broader economic context by examining the effect of possible stochastic parameters. As compared to the Black-Scholes market, the parameters describing the risky assets in Merton (1973) are allowed to be dependent on a stochastically evolving state variable. This broader framework leads to the extension of the mutual fund separation theorem, namely: the three-fund separation theorem. When allowing for stochasticity, the optimal portfolio no longer only consists of a position in the risky asset and a mutual fund, instead an extra position in the assets is needed so that the agent can hedge itself against fluctuations in the stochastic parameters. Thus, the optimal portfolio in the extended framework not only consists of the mean-variance optimal demand, which achieves the highest Sharpe ratio, but also consists of hedging demand. Intuitively the hedging demand is explained as the wish to secure future wealth and/or consumption. This wish can be realized by taking positions in assets that are correlated to the variables that determine the future evolution of the risky assets (Merton, 1973). As the models considered in this thesis allow for more stochasticity than the Black-Scholes market, the agent's need to hedge themselves against this stochasticity will also come back in the optimal investment strategies we consider in this thesis[3]. A final important remark on the three-fund separation theorem is that the optimal relative portfolio now differs per agent. Hence, this differs from the framework with constant parameters where the optimal relative allocation to the risky assets is the same for each agent. This difference can be explained by the fact that the degree to which assets are suitable for hedging is dependent on the agent's risk preferences (Merton, 1973).

Although the work of Merton (1969, 1971, 1973) gives an introduction to continuous time portfolio optimization and the mechanisms behind possible stochastic parameters, the (welfare) effects of stochastic interest, inflation, and volatility are still to be analyzed in more detail.

1.2 Incomplete markets and the Martingale method

This section focuses on the method we use to find the optimal investment strategy and how it relates to the method used by Merton (1969). Both in this thesis and in Merton's work one is interested in maximizing expected utility subject to a budget constraint. When an agent receives utility from terminal wealth alone in the Black-Scholes market, such an optimization problem would look as follows:

$$\begin{aligned} &\sup_{\theta} \ \mathbb{E}\Big[U(W_T)\Big] \\ &\text{s.t. } dW_t = W_t(r + \theta_t(\mu - r)) + W_t\theta_t\sigma dZ_t \end{aligned} \tag{1.3}$$

where W_t is the agent's wealth at time t. Hence, W_T is the agent's terminal wealth, i.e. the wealth at the retirement date. θ_t is the fraction of wealth invested in the risky asset at time t, Z_t a standard Brownian motion driving the uncertainty in the risky asset, μ the mean return of the stock, σ the volatility of the stock, and r the risk-free rate. The budget constraint in (1.3) states that the agent earns the risk premium $\mu - r$ on the fraction of wealth that is invested in the risky asset. The rest of its wealth is invested in the riskless asset, which has a rate of return equal to r.

An optimization problem as defined above is typically called a dynamic portfolio optimization problem since the agent faces a budget constraint that is dynamically changing over time. In such a dynamic portfolio problem the agent chooses its investment strategy θ_t optimally for each t, leading to the

[3]The strategies we find in this thesis include mean-variance optimal demand and hedge demands. The mean-variance optimal demand will be inversely related to the agent's risk aversion parameter. The hedge demands generally contain a non-linear relationship with the agent's risk aversion parameter.

optimal terminal wealth (Munk, 2017). Merton (1969, 1971, 1973) solves the dynamic optimization problem using a dynamic programming method. With a dynamic programming method, the optimization problem is solved via its recursive character; investments made today will impact the financial situation of the agent in the future. Utilizing this recursive structure will ultimately lead to a nonlinear PDE, called the Hamilton-Jacobi-Bellman (HJB) equation (Munk, 2017). The construction and verification of answers to the HJB equation typically require verification theorems[4]. In general this leads to complex problems, specifically when constraints on the control variables are imposed (Cox and Huang, 1989, 1991).

Therefore, Karatzas et al. (1987) and Cox and Huang (1989, 1991) proposed the Martingale method, which overcomes the difficulties with verifying solutions. The core of this new method is the replacement of the dynamic constraint with a static constraint so that the portfolio optimization problem can be solved via its Lagrangian. Underlying this replacement lies the idea that, in a complete market, wealth can be viewed as a traded asset. As such, the initial value of the optimal wealth can be determined using replication arguments similar to how one would, for example, determine the price of a call option in the Black-Scholes market (Munk, 2017). Hence, from the replication arguments it follows that the static constraint is such that the discounted optimal terminal wealth in expectation equals the initial wealth (Munk, 2017; Schumacher, 2020). The biggest advantage of the Martingale method over the dynamic programming method is that it takes care of the verification of the constraints in itself (Cox and Huang, 1989, 1991) and thus no additional verification methods are needed.

For the example with the portfolio optimization problem in the Black-Scholes market in (1.3), the new static optimization problem looks as follows:

$$\begin{aligned} &\sup_{W_T} \ \mathbb{E}\Big[U(W_T)\Big] \\ &\text{s.t. } \mathbb{E}\Big[\phi_T W_T\Big] = W_0 \end{aligned} \tag{1.4}$$

where ϕ_T is the market's pricing kernel, which in this case is used to discount the value of terminal wealth. In the Black-Scholes market the dynamics of the pricing kernel are as follows:

$$\frac{d\phi_t}{\phi_t} = -rdt - \lambda dW_t \tag{1.5}$$

Here $\lambda = \frac{\mu - r}{\sigma}$ is the market price of risk of the Brownian motion that drives the uncertainty in the stock. Note that compared to the dynamic problem, both the control variable and the budget constraint have changed in the static opti-

[4]For an extensive overview of verification theorems for the HJB equation we refer to Korn and Kraft (2002), Korn and Kraft (2004), Kraft (2005), and Dybvig and Liu (2011).

mization problem. The modification of the budget constraint can be explained from the replication argument; when the expected discounted terminal wealth is strictly larger than the initial endowment, it is not possible to find a portfolio that replicates the terminal wealth. If the discounted expected terminal wealth is less than the initial endowment, it is possible to find a portfolio that replicates the terminal wealth. However, extra utility can then be retrieved for the terminal wealth that exactly replicates the initial endowment. Thus, the terminal wealth needs to be found that, if discounted, exactly equals the initial endowment in expectation. An intuitive argument for the replacement of the control variable is that the dynamic optimization problem corresponds to the real world in the sense that the agent chooses its optimal investment strategy, which dynamically leads to the optimal wealth. When applying the Martingale method, one does not solve for the optimal investment strategy but from the replication arguments it follows that the agent directly optimizes its terminal wealth. Since the optimal terminal wealth alone is sufficient to find the corresponding optimal strategy, the problem does not require extra control variables. We thus see that the portfolio that leads to the optimal terminal wealth still has to be found once the optimal wealth is known. In this thesis we will link an investment strategy to the optimal wealth by exploiting the martingale property of the product of the optimal wealth process and the pricing kernel, from which we know it has to be a martingale by the Fundamental Theorem of Asset Pricing (FTAP) (Schumacher, 2020).

The core of the Martingale method is thus the possibility to replicate optimal terminal wealth with a self-financing portfolio. For this replication argument to be valid, one should act in a complete financial market. In an incomplete market infinitely many pricing kernels exist; the pricing kernel used to discount the optimal terminal wealth in the static budget constraint will thus be non-unique (Schumacher, 2020). Consequently, it will be unclear which strategy has led to the optimal wealth. The financial market models we consider in Chapter 2, Chapter 3, and Chapter 4 will be incomplete. In other words, the markets considered give the opportunity to invest in less assets than there are risk drivers. Note that this constitutes a difference from the setup of Merton (1969, 1971, 1973) where the number of risky assets the agent can invest in exactly equals the number of Brownian motions driving the market. Hence, to apply the Martingale method, we propose a further refinement of our methodology. We do so by introducing the abstract distinction between the *original* financial market and its *fictional* counterpart[5], introduced by Karatzas et al. (1991), Cvitanic and Karatzas (1992), Xu and Shreve (1992), and Kamma (2023). In the context of this thesis, the original financial market is the incomplete market in which the agent acts. In the fictional market we allow for trading in one or more extra assets so that it

[5]Throughout this thesis we will use the terms original/incomplete/primal market and fictional/artificial/complete/dual market interchangeably.

becomes possible to trade in the previously untraded sources of risk. Hence, a complete market is created artificially. In the context of a real-life pension fund, one could interpret the original market as the situation the fund faces due to certain constraints, possibly induced by law. An example is that the Dutch government does not issue inflation-indexed bonds (Pelsser, 2019) and thus pension funds cannot hedge themselves directly against Dutch inflation[6]. On the other hand, the fictive market is the market the fund would face in which no constraints exist, and thus investments in any possibly traded asset could be made, i.e. the fund can invest in Dutch inflation-indexed bonds.

The artificially created complete market enables us to solve the portfolio optimization problem with the Martingale method. However, the resulting investment strategy will generally allocate part of the agent's wealth to the assets that have been added to arrive at a complete market setup. As a consequence, to ensure that the strategies are admissible in the original market, they must be modified such that no investment is made in the added asset(s). In the literature, the link between a strategy in the original and fictional market finds its origins in the context of primal and dual optimization problems. The duality perspective mathematically formalizes why the method of artificially completing a market leads to the optimum in the incomplete market. The relationship between the primal and dual market can be interpreted as follows: an agent initially faces a primal optimization problem which, due to constraints, cannot be solved directly. In our thesis this is due to the non-uniqueness of the static budget constraint. Under certain regularity conditions the optimization problem can be reformulated to its dual form, which allows one to deal with the constraints more easily so that the answer to the original problem can be found. Consequently, we have found a new optimization problem, the so-called dual problem. The transition to the dual problem is thus driven by lifting the trading constraints in the primal market. This is an important principle underlying the primal-dual approach; the incompleteness in the primal market is not motivated by the non-existence of assets but by trading constraints which make it impossible to trade in certain assets. The primal and dual market thus contain the same assets; only in the primal market it is not possible to trade in certain assets, making the market incomplete.

It remains to be answered how we can make the optimal dual strategy admissible in the primal market. According to the duality theory proposed by Karatzas et al. (1991), Cvitanic and Karatzas (1992), Xu and Shreve (1992), and Kamma (2023), the prices of risk in the dual market of the Brownian motions untraded in the primal market are equal to the sum of the prices of risk in the primal market and an extra term, henceforth the perturbation term. The existence of a perturbation term is thus the only difference between the

[6] Although the Dutch government does not issue inflation-indexed bonds, several other European countries do, for example France (Pelsser, 2019). Current research shows a significant rise in the inflation-linked bond market (Garcia and van Rixtel, 2007), enhancing the ability to hedge against inflation on a European level.

primal and dual market, the asset dynamics in the dual market are as given in the primal market. The trading constraints in the primal market completely determine which prices of risk in the dual market are influenced by the perturbation term. In other words, the structure of the dual market is completely determined by the primal market and the trading constraints. After solving the dual optimization problem, the perturbation term is chosen such that no investments in the untraded assets are made. The perturbation term is thus determined as part of the dual problem. From a mathematical perspective, this choice of the perturbation term corresponds to minimizing the dual objective, which is equivalent to solving the primal optimization problem. In other words, the perturbation term that reduces the demand of the untraded assets to zero can be interpreted as the shadow price that links the Lagrangian of the dual problem to the Lagrangian of the primal problem. Hence, artificial completion of the market in combination with choosing the appropriate perturbation term allows us to uniquely determine the static budget constraint. Using a modified price of risk for the untraded Brownian motions in the dual optimization problem can thus be seen as the method that allows us to satisfy the trading constraints in the original market. Hence, in the context of duality the sole purpose of the perturbation term is to link the solution of the dual problem to the solution of the primal problem. For firm mathematical results on why the duality between the incomplete market and its complete reformulation holds, we refer to Karatzas et al. (1991), Cvitanic and Karatzas (1992), Xu and Shreve (1992), and Kamma (2023).

We aim to deepen the understanding of the primal-dual framework by presenting an alternative approach to the primal-dual approach to solve the portfolio optimization problem. The fictitious completion method can be explained from the fact that an incomplete market is represented by infinitely many pricing kernels (Schumacher, 2020). Hence, an artificial complete market can be created by adding assets, so that the Martingale method can be applied. In this artificial market an optimal investment strategy can be found. Finally, the price of risk of the Brownian motions that are untraded in the original market can be chosen such that no investments in the added assets are made. This particular choice of the price of risk pins down a pricing kernel in the incomplete market, and thus the solution in the incomplete market is found. The choice of a particular pricing kernel can be understood as the equivalent of choosing the perturbation term in the primal-dual approach. However, the fictitious completion method forces one to change the structure of the financial market after optimizing; a choice for a particular pricing kernel will influence the market dynamics. Thus, the fictitious completion approach forces an ex-post interruption in the financial market. Such an interruption is not needed when using the primal-dual approach. Hence, we argue that the primal-dual approach is a methodological refinement of the fictitious completion approach. Although the fictitious completion approach differs from the primal-dual approach in how market incompleteness is dealt with, in general the two methods will lead to the same outcome. This is the case for the Heston (1993) stochastic volatility model discussed in Chapter 3. However, the KNW model discussed in Chapter 2 and the CP2022 model discussed in Chapter 4 have a more complex structure in which the alternative approach causes problems[7]. For example in the KNW model, the fictitious completion view leads to theoretical problems since the price of risk of the Brownian motion that is untraded in the original market indirectly influences the volatility matrix of the market. Any investment strategy in the KNW model will depend on the volatility matrix. Hence, the optimal strategy will implicitly depend on the price of risk of the untraded Brownian motion. When applying the fictitious completion approach here, one would, after optimization, choose the price of risk of the untraded Brownian motion such that no investment in the added asset is made. However, for this particular choice of the market price of risk the volatility matrix will implicitly change. Consequently, the previously found optimum is no longer optimal. We see that a circularity problem arises and thus a more nuanced approach is needed. This is where the primal-dual approach comes in. The primal-dual approach fixes the market dynamics ex-ante since incompleteness is explained by trading constraints. This prevents one from making ex-post changes to the original market dynamics so that, for example, a numerical implementation of the optimum becomes possible.

To summarize this section; we have intuitively explained the switch from the dynamic optimization problem to its static reformulation. To solve the static optimization problem with the Martingale method in an incomplete market, we have introduced the distinction between the original and artificial market. In this thesis we deal with the original and artificial market from a primal-dual perspective. The primal-dual approach allows us to solve a dual optimization problem, which is linked to the primal optimization problem via the use of a perturbation term. This method explains market incompleteness from the perspective of trading constraints. Consequently, the original market specifications are fixed before optimizing, which prevents us from changing the market dynamics after optimizing.

1.3 Welfare comparison

AZL currently generates life cycles according to the investment strategy that is optimal in the Black-Scholes market, whereas the economic parameters are generated by the Commissie Parameters (2022) model. Consequently, a strategy that is suboptimal within the model assumed to represent the economy is used, resulting in welfare losses. One of the goals of this thesis is to estimate the welfare effects of using the investment strategy found by Merton

[7]Since the alternative method cannot be used in the KNW and CP2022 model, we choose to also present the results of the Heston (1993) stochastic volatility model in Chapter 3 according to the primal-dual logic needed to find the optimum in the other models.

(1969), if it is assumed that the true economy is represented by a more comprehensive financial market model than the Black-Scholes model. To perform this welfare comparison, we first have to find the optimal investment strategy in the model of interest. Once the analytical investment formula is found, we should indicate which part of this formula represents the mean-variance optimal investment strategy, representing Merton's formula. Note that by the three-fund separation theorem discussed in Section 1.1 the optimal investment strategies discussed in this thesis will be a composition of hedging demand and the mean variance optimal demand. As such it is known that Merton's investment strategy can be identified in the optimal strategies considered in this thesis. We state that, depending on the model, the choice of how to explicitly specify the mean-variance optimal demand might be subject to debate. We present this discussion in the section devoted to that particular model.

When both the optimal and the suboptimal investment strategy are determined, a welfare comparison between these strategies can be performed. In line with literature[8], we perform this welfare comparison on the basis of the certainty equivalent (CE). Formally, the certainty equivalent of random terminal wealth W_T is defined according to the following formula (Weber, 2001):

$$U(\text{CE}) = \mathbb{E}\Big[U(W_T)\Big] \tag{1.6}$$

The certainty equivalent can thus be interpreted as the constant terminal wealth that leads to the same expected utility as the random terminal wealth W_T. Although an analytical distribution of terminal wealth might exist in the models we consider, we choose to calculate certainty equivalents on the basis of a numerical implementation of the optimal strategy. We argue that this deepens our understanding of the optimal strategy more than presenting additional analytical results. Furthermore, this helps us avoid technicalities when finding the distribution of terminal wealth. Hence, after we have analytically found an optimal strategy, we simulate m scenarios of our financial market over the time horizon T. If we let $W^*_{T,i}$ be the terminal wealth in scenario i following from investing according to the optimal strategy θ^*_t at each time point for an agent with risk aversion parameter γ, we estimate the numerical certainty equivalent of this strategy over m simulations as[9]:

$$\text{CE} = \left((1-\gamma)\frac{1}{m}\sum_{i=1}^{m} U(W^*_{T,i})\right)^{\frac{1}{1-\gamma}} \tag{1.7}$$

We can then define the percentage welfare loss of incorrectly using Merton's mean variance optimal strategy instead of the strategy optimal in the model

[8]See for example Castañeda and Reus (2019) or Balter and Schweizer (2024), among others.

[9]The transition from the analytical CE in (1.6) to the numerical CE in (1.7) is motivated by the law of large numbers.

of interest as:

$$\frac{\mathrm{CE}_{\mathrm{Merton}} - \mathrm{CE}_{\mathrm{opt}}}{\mathrm{CE}_{\mathrm{opt}}} \tag{1.8}$$

where $\mathrm{CE}_{\mathrm{Merton}}$ and $\mathrm{CE}_{\mathrm{opt}}$ are the certainty equivalents following from the mean variance optimal and true optimal strategy, respectively. We will present this welfare loss for different values of γ to investigate the dependence of the welfare loss on the level of risk aversion of the agent.

Another fundamental aspect influencing the life cycles generated by AZL is the risk preference parameter. Currently, the risk preference parameter of a participant is estimated via the risk preference research. Consequently, a crucial question underlying this method is how inaccurate estimates of the risk preference parameter influence the participants' welfare. Therefore, we also present welfare losses following from investments made according to the wrong value of γ. To this end, we distinguish the true risk parameter of an agent, γ_{true}, from the risk parameter derived from the risk preference research, $\gamma_{\mathrm{measured}}$. When $\gamma_{\mathrm{measured}} = \gamma_{\mathrm{true}}$, an agent invests optimally, and thus no welfare loss arises. In any other case, the agent does not follow its optimal strategy, resulting in welfare losses. To quantify the size of these welfare losses, it is important to clarify how the corresponding certainty equivalents are calculated. The CE following from γ_{true} can be calculated directly from the formula in (1.7). However, for $\gamma_{\mathrm{measured}}$ one has to be more careful. In this case we calculate the value of terminal wealth on the basis of the wrong strategy determined by $\gamma_{\mathrm{measured}}$. On the other hand, evaluating how an agent perceives this wrong strategy should be done on the basis of the agents true risk preference parameter. Linking this to the formula in (1.7), we find that the values of $W_{T,i}$ are found on the basis of $\gamma_{\mathrm{measured}}$. However, the utility function and certainty equivalent are evaluated on the basis of γ_{true}. Consequently, the percentage welfare loss of using a wrong risk preference parameter can be determined by combining the two certainty equivalents:

$$\frac{\mathrm{CE}_{\mathrm{measured}} - \mathrm{CE}_{\mathrm{true}}}{\mathrm{CE}_{\mathrm{true}}} \tag{1.9}$$

where $\mathrm{CE}_{\mathrm{measured}}$ and $\mathrm{CE}_{\mathrm{true}}$ are the certainty equivalents following from investing according to $\gamma_{\mathrm{measured}}$ and γ_{true}, respectively. We will present welfare losses for different pairs of $\gamma_{\mathrm{measured}}$ and γ_{true}.

We will also present the standard errors belonging to the welfare losses. To find an estimate for the standard error of a simulated certainty equivalent following from the strategy θ_t^x, we define $\mathbb{E}[U(W_T^x)]$ and $\mathrm{Var}[U(W_T^x)]$ as the theoretical mean and variance of the utility derived from terminal wealth resulting from this strategy. Furthermore we let $\bar{U}(W_T^x) = \frac{1}{m}\sum_{i=1}^{m} U(W_{T,i}^x)$ and $f(x) = \Big((1-\gamma)x\Big)^{\frac{1}{1-\gamma}}$, i.e. $\bar{U}(W_T^x)$ is the sample mean of utility derived from terminal wealth over m simulations and $\mathrm{CE}_x = f(\bar{U}(W_T^x))$. We can then use the delta method in combination with the central limit theorem to find the asymptotic distribution of CE_x:

$$\sqrt{m}\Big(f(\bar{U}(W_T^x)) - f(\mathbb{E}[U(W_T^x)])\Big) \xrightarrow{D} \mathcal{N}\Big(0, f'(U(W_T^x))^2 \operatorname{Var}[U(W_T^x)]\Big) \quad (1.10)$$

Based on this asymptotic distribution, we propose the following estimate for the standard error of CE_x:

$$\begin{aligned} \text{SE}\left(\text{CE}_x\right) &\approx \left|f'(\bar{U}(W_T^x))\right| \frac{\text{STD}\left(U(W_T^x)\right)}{\sqrt{m}} \\ &= \left|\frac{\left((1-\gamma)\bar{U}(W_T^x)\right)^{\frac{1}{1-\gamma}}}{(1-\gamma)\bar{U}(W_T^x)}\right| \frac{\text{STD}\left(U(W_T^x)\right)}{\sqrt{m}} \end{aligned} \quad (1.11)$$

where $\text{STD}\left(U(W_T^x)\right)$ is the sample standard deviation of utility from terminal wealth, i.e. $\text{STD}\left(U(W_T^x)\right) = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(U(W_{T,i}^x) - \bar{U}(W_T^x)\right)^2}$.

Finally, we are interested in the standard error of the welfare loss that follows from using a suboptimal strategy θ_t^y instead of the optimal strategy θ_t^x. Therefore we define the multivariate mean and variance covariance matrix of utility derived from the terminal wealth that follows from strategies θ_t^x and θ_t^y as $\mathbb{E}[P] \equiv \begin{bmatrix}\mathbb{E}[U(W_T^x)] & \mathbb{E}[U(W_T^y)]\end{bmatrix}$ and Σ_P, respectively. We then define the multivariate sample mean as $\bar{P} = \begin{bmatrix}\bar{U}(W_T^x) & \bar{U}(W_T^y)\end{bmatrix}$ and $f(x,y) = \left(\frac{x}{y}\right)^{\frac{1}{1-\gamma}} - 1$. Hence, the welfare loss of following the suboptimal strategy θ_t^y instead of the optimal strategy θ_t^x is defined as $f\Big(\bar{U}(W_T^x), \bar{U}(W_T^y)\Big) \equiv f(\bar{P})$. Consequently, we can apply the delta method in combination with the central limit theorem to find the asymptotic distribution of the welfare loss:

$$\sqrt{m}\Big(f(\bar{P}) - f(\mathbb{E}[P])\Big) \xrightarrow{D} \mathcal{N}\Big(0, J(P)'\Sigma_P J(P)\Big) \quad (1.12)$$

where J is the Jacobian with first order derivatives, i.e. $J(P) = \begin{bmatrix}\frac{\partial f}{\partial U(W_T^x)} \\ \frac{\partial f}{\partial U(W_T^y)}\end{bmatrix}$. Inspired by the asymptotic distribution of $\frac{\text{CE}_\text{x} - \text{CE}_\text{y}}{\text{CE}_\text{y}}$, we propose the following estimate for the standard error of the welfare loss:

$$\text{SE}\left(\frac{\text{CE}_x - \text{CE}_y}{\text{CE}_y}\right) \approx \sqrt{\frac{J(\bar{P})'\Sigma_P^{x,y} J(\bar{P})}{m}} \quad (1.13)$$

Here, $\Sigma_P^{x,y}$ is the in-sample variance covariance matrix of the simulated utility values, $J_1(\bar{P}) = \frac{1}{1-\gamma}\frac{\bar{U}(W_T^x)^{\frac{1}{1-\gamma}-1}}{\bar{U}(W_T^y)^{\frac{1}{1-\gamma}}}$ and $J_2(\bar{P}) = -\frac{1}{1-\gamma}\frac{\bar{U}(W_T^x)^{\frac{1}{1-\gamma}}}{\bar{U}(W_T^y)^{\frac{1}{1-\gamma}+1}}$.

Chapter 2

Stochastic interest rate

Until, 2022 the scenario sets published by DNB were generated according to the KNW model, which is introduced in the literature by Koijen et al. (2010). Consequently, it precedes the model by Commissie Parameters (2022), which is currently used to generate scenario sets. It is therefore a natural choice to examine the welfare effects of stochastic interest rates by using the KNW model. The model of Koijen et al. (2010) accommodates uncertainty about interest rates, expected inflation, unexpected inflation, and stock returns. Furthermore, it allows for time-varying risk premia. Hence, the model by Koijen et al. (2010) enriches the framework of Merton (1969) from multiple perspectives. In Section 2.1, we will present the KNW financial market, combined with the dynamics of an inflation-indexed bond in which the agent cannot invest in the primal market. In Section 2.2, we will solve the dual optimization problem and link the corresponding investment strategy to a strategy in the original incomplete market. In Section 2.3, we will exploit the properties of the optimal strategy using a numerical analysis. Finally, in Section 2.4 we will provide the welfare losses of using Merton's suboptimal mean-variance strategy and the welfare effects of using a wrong risk preference parameter.

Koijen et al. (2010) numerically estimate allocations in a life cycle framework in which households receive labor income. Hence, no analytical optimal investment strategy in the (in)complete market is found. They primarily focus on the effect of time variation in risk premia on the life cycle. Consequently, the welfare effects of suboptimal strategies as we discuss in this thesis is not investigated by Koijen et al. (2010). The KNW financial market model is closely related to the models proposed by Campbell and Viceira (2001), Brennan and Xia (2002), and Sangvinatsos and Wachter (2005), who all propose a financial market in which the term structure is described by stochastic interest rates and inflation. We deviate from these papers when solving the portfolio optimization problem by using the primal-dual approach, and thus an alternative method is used to deal with market incompleteness.

Campbell and Viceira (2001) set up a model in discrete time with constant risk premia[10]. They find that the optimal portfolio consists of a combination of mean-variance optimal demand and hedging demand, in line with the findings on the three-fund separation theorem by Merton (1973). Campbell and Viceira (2001) specifically focus on the composition of hedging demand; nominal bonds are found to be most suitable for hedging interest rate risk, whereas inflation-linked bonds are the most suitable instrument to hedge against inflation risk (Campbell and Viceira, 2001). No results on the welfare effects of suboptimal strategies are presented.

Similarly to Koijen et al. (2010), Brennan and Xia (2002) define a two factor model for the term structure of interest rates in continuous time. However, Brennan and Xia (2002) do not allow for time variation in risk premia. In contrast to Merton (1969), Brennan and Xia (2002) use the Martingale method to solve the optimization problem, rather than the dynamic programming method. Brennan and Xia (2002) deal with market incompleteness by stating that the optimal proportional allocation of wealth is independent of the untraded Brownian motion. As this merely provides a heuristic argument for the effect of market incompleteness on the optimal portfolio, we extend this framework from a mathematical point of view by using the primal-dual approach. As it turns out, this mathematical rigid method leads to similar optimal investment strategies. Brennan and Xia (2002) find an analytical optimal investment strategy in the incomplete market on the basis of this heuristic argument, for agents that receive utility from terminal wealth and/or consumption. An analytical formula for the gain in certainty equivalent when investing optimally versus investing according to a myopic strategy is presented. However, no extensive analysis of this formula or numerical results on the effects of suboptimal investment are presented.

Sangvinatsos and Wachter (2005) also define a continuous time model that takes stochastic interest rates and inflation into account. In contrast to Campbell and Viceira (2001) and Brennan and Xia (2002), they allow for time variation in risk premia, leading to term structures that are described by three factors instead of two factors. Sangvinatsos and Wachter (2005) deal with market incompleteness by choosing the particular pricing kernel that reduces the demand of the added assets to zero. This is the method we introduce as the fictitious completion approach in Section 1.2. In the KNW model this approach causes problems due to the dependence of the variance covariance matrix on the prices of risk of the untraded Brownian motion. Hence, we argue that we propose a methodological refinement of the method used by Sangvinatsos and Wachter (2005) to find the optimal strategy. Although this constitutes a difference between our approach and the approach of Sangvinatsos and Wachter (2005), similar strategies are found. Furthermore, Sangvinatsos and Wachter

[10] Since the model of Campbell and Viceira (2001) is defined in discrete time, we will only compare the optimal strategy in the KNW model to the strategies found by Brennan and Xia (2002) and Sangvinatsos and Wachter (2005).

(2005) present results on the utility cost of suboptimal investment. Although this analysis is performed from an analytical perspective, they find significant welfare effects of using suboptimal strategies. In particular, neglecting the time variation in risk premia leads to significant welfare losses (Sangvinatsos and Wachter, 2005). This is in line with the findings we will present in Section 2.4.

Also outside the context of the KNW model the effect of interest rate risk on portfolio selection has been studied extensively. We briefly present some important work: Munk et al. (2004) deviates from the setup of Campbell and Viceira (2001), Brennan and Xia (2002), Sangvinatsos and Wachter (2005), and Koijen et al. (2010) by allowing a mean reverting process for the excess return on stocks. They ultimately find that the extra randomness introduces extra hedging demand. No welfare effects of suboptimal investment are presented. Also the effect of stochastic interest rates without the existence of stochastic inflation has been studied, for example by Sørensen (1999), Brennan and Xia (2000), and Munk and Sørensen (2004), all in contexts where no welfare losses have been discussed. All papers discussed thus far have in common that the static optimization problem is solved rather than the dynamic problem. For important findings on the setup of the HJB equation in a stochastic interest rate environment we refer to Korn and Kraft (2002).

2.1 Financial market

Koijen et al. (2010) postulate a two factor model for the term structure of interest rates. The two factors are the latent state variables, which also determine the evolution of the stochastic instantaneous nominal interest rate and the expected inflation. In the original market possible investments can be made in a nominal money market account (MMA) paying the instantaneous nominal risk free rate, a nominal stock, and two nominal bonds with different maturities (Koijen et al., 2010). The randomness in the term structure of interest is determined by the two dimensional state variable X_t, whose dynamics are given as follows:

$$\begin{aligned} dX_t &= -K_X X_t dt + [I_{2\times 2} \quad 0_{2\times 2}] dZ_t \\ &= -K_X X_t dt + \Sigma_X dZ_t \end{aligned} \tag{2.1}$$

where $I_{2\times 2}$ is the 2 dimensional identity matrix, $0_{2\times 2}$ the 2 dimensional null matrix, $Z_t \in \mathbb{R}^{4\times 1}$ a 4 dimensional vector with independent Brownian motions and, $X_0 = [0\ 0]'$. The remaining quantities in (2.1) therefore possess the following dimensions: $X_t \in \mathbb{R}^{2\times 1}$, $K_X \in \mathbb{R}^{2\times 2}$ and $\Sigma_X \in \mathbb{R}^{2\times 4}$. We see that the state variable X_t is modeled according to a multidimensional Ornstein-Uhlenbeck process, which is mean reverting around 0 with mean reversion speed K_X. Furthermore the structure of Σ_X implies that only the first two Brownian motions in Z_t influence the development of X_t. Consistent with Koijen et al. (2010) we impose K_X to be a lower triangular matrix.

The evolution of the instantaneous nominal interest rate r_t and instantaneous expected inflation π_t are driven by the state variables:

$$\begin{aligned} r_t &= \delta_{0,r} + \delta_{1,r}' X_t \\ \pi_t &= \delta_{0,\pi} + \delta_{1,\pi}' X_t \end{aligned} \tag{2.2}$$

where $\delta_{0,r} > 0$, $\delta_{0,\pi} > 0$, $\delta_{1,r} \in \mathbb{R}^{2\times1}$, and $\delta_{1,\pi} \in \mathbb{R}^{2\times1}$.

To accommodate a link between the nominal and real quantities in this financial market, a commodity price index (CPI) Π_t is postulated:

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi' dZ_t \tag{2.3}$$

where $\sigma_\Pi \in \mathbb{R}^{4\times1}$ and $\Pi_0 = 1$. In line with Koijen et al. (2010), we assume that $\sigma_{\Pi,4} = 0$. The CPI represents the realized (unexpected) rate of inflation, being driven by the expected rate of inflation π_t and influenced by the randomness coming from Z_t. To understand the role of the CPI, we focus on the role of market (in)completeness in Koijen et al. (2010)'s model. As investments can only be made in nominal assets, an agent can only hedge itself against changes in the nominal economy but not against changes in the real economy. Hence, the agent can only hedge itself against the expected inflation but not against the realized inflation Π_t. In other words, investments can only be made in a nominal MMA, nominal stock, and two nominal bonds and thus the market is driven by more sources of risk than the agent can invest in, leading to an incomplete market. When estimating the model parameters, Koijen et al. (2010) deal with the market incompleteness by imposing that the price of risk of the untraded Brownian motion equals zero. This suggests that realized inflation risk does not influence expected returns. We extend the model with an inflation-linked bond so that we can solve the portfolio optimization problem with the Martingale method via the primal-dual approach. In the original market an agent cannot invest in this inflation-linked bond, so that we maintain market incompleteness. In the dual market we lift the trading constraint and thus all sources of risk that influence Π_t can be traded. The primal-dual approach is based on the idea that the inflation-indexed bond exists in the primal market, but the agent cannot invest in it due to trading constraints. This logic forces us to fix the price of risk of the Brownian motion untraded in the original market, which fixes the dynamics of the inflation-indexed bond. We assume that the price of risk of the Brownian motion untraded in the original market equals zero, in line with how Koijen et al. (2010) deal with market incompleteness when estimating the model parameters. In the dual market, we will then choose the perturbation term corresponding to this price of risk so that the demand of the inflation-indexed bond is reduced to zero, which corresponds to optimizing the primal objective.

We then move to the specifications of the financial instruments. First of all the agent can invest in a nominal money market account, whose dynamics are given as follows:

$$\frac{dB_t}{B_t} = r_t dt \tag{2.4}$$

where $B_0 = 1$. Furthermore, the agent can invest in a nominal stock, described by the following stochastic differential equation (SDE):

$$\begin{aligned}\frac{dS_t}{S_t} &= (r_t + \eta_S)dt + \sigma_S' dZ_t \\ &= (r_t + \sigma_S' \Lambda_t)dt + \sigma_S' dZ_t\end{aligned} \tag{2.5}$$

where $\eta_S \in \mathbb{R}$ is the constant risk premium of the stock, $\sigma_S \in \mathbb{R}^{4\times 1}$, and $S_0 = 1$. Λ_t is the time varying price of risk, who Koijen et al. (2010) assume to be a linear function of the state variables:

$$\Lambda_t = \Lambda_0 + \Lambda_1 X_t \tag{2.6}$$

where $\Lambda_0 \in \mathbb{R}^{4\times 1}$ and $\Lambda_1 \in \mathbb{R}^{4\times 2}$. As described above, the primal-dual approach forces the price of risk of the untraded Brownian motion in the original market to be fixed. We therefore assume that $\Lambda_{0,3} = \Lambda_{1,(3,1)} = \Lambda_{1,(3,2)} = 0$. Note that the assumption that the equity risk premium in (2.5) is constant, imposes the constraint $\sigma_S'\Lambda_t = \eta_S$. Kamma and Pelsser (2022) show that the constraint implies that $\sigma_S'\Lambda_0 = \eta_S$ and $\sigma_S'\Lambda_1 = 0_2'$. Combining the assumption of the price of risk of the untraded Brownian motion with parameter estimates for the prices of risk of the first two Brownian motions thus leads to a fixed equity risk premium η_S.

To link possible investment strategies to payouts, the use of a pricing kernel is a fundamental concept in quantitative finance (Schumacher, 2020). The pricing kernel is a stochastic process that allows financial assets to be priced, exploiting the no-arbitrage condition. By virtue of using the money market account B_t as numéraire, Koijen et al. (2010) postulate the following *nominal* pricing kernel:

$$\frac{d\phi_t^N}{\phi_t^N} = -r_t dt - \Lambda_t' dZ_t \tag{2.7}$$

where $\phi_0^N = 1$. Using the nominal pricing kernel from (2.7), we define the real pricing kernel as $\phi_t^R = \phi_t^N \Pi_t$. The dynamics of ϕ_t^R are then as follows (for derivations see Appendix A.1):

$$\begin{aligned}\frac{d\phi_t^R}{\phi_t^R} &= -(r_t - \pi_t + \sigma_\Pi'\Lambda_t)dt - (\Lambda_t' - \sigma_\Pi')dZ_t \\ &= -R_t dt - \Lambda_t^{R\prime} dZ_t\end{aligned} \tag{2.8}$$

where $\phi_0^R = 1$. We define R_t as the instantaneous real interest rate and $\Lambda_t^R =$

$\Lambda_t - \sigma_\Pi$ as the real pricing kernel. Note that R_t is specified as follows:

$$\begin{aligned} R_t &= r_t - \pi_t + \sigma_\Pi' \Lambda_t \\ &= (\delta_{0,r} - \delta_{0,\pi} + \sigma_\Pi' \Lambda_0) + (\delta_{1,r} - \delta_{1,\pi} + \Lambda_1' \sigma_\Pi)' X_t \\ &= R_{0,R} + R_{1,R}' X_t \end{aligned} \tag{2.9}$$

Next to the money market account and stock, the agent can invest in two nominal bonds with different maturities. To find the price dynamics of the nominal bonds, we rely on the work of Duffie and Kan (1996). Duffie and Kan (1996) find that bond prices are an exponentially affine function of time and the state variables under the following conditions: (i) the short rate is affine in the state variables and (ii) the state variables are described by a parametric multivariate Markov diffusion process. As the definitions of the state variable and the short rate in (2.1) and (2.2) meet these conditions, the exponentially affine structure of the bond prices can be used. We denote the price of a nominal bond maturing at time $t+\tau_i$ by $P^N_{t+\tau_i}(t, X_t) = \mathbb{E}\left[\frac{\phi^N_{t+\tau_i}}{\phi^N_t} \middle| \mathcal{F}_t\right] = \exp\left(A^N(\tau_i) + B^N(\tau_i)' X_t\right)$ for $i = 1, 2$ with $\tau_1 \neq \tau_2$. A key observation in deriving the dynamics of $P^N_{t+\tau_i}(t, X_t)$ is that the product of the bond prices and the pricing kernel must be a martingale by FTAP (Schumacher, 2020). Exploiting the martingale property of $\phi^N_t P^N_{t+\tau_i}(t, X_t)$ by the use of Itô calculus leads to the following dynamics for nominal bond prices (for detailed derivations see Appendix A.1):

$$\frac{dP^N_{t+\tau_i}(t, X_t)}{P^N_{t+\tau_i}(t, X_t)} = \left(r_t + B^N(\tau_i)' \Sigma_X \Lambda_t\right) dt + B^N(\tau_i)' \Sigma_X dZ_t \tag{2.10}$$

The functions $A^N(\tau_i) \in \mathbb{R}$ and $B^N(\tau_i) \in \mathbb{R}^{2\times 1}$ are specified as follows:

$$\begin{cases} A^N(\tau_i) = \int_0^{\tau_i} -\delta_{0,r} + \frac{1}{2} B^N(s)' \Sigma_X \Sigma_X' B^N(s) - B^N(s)' \Sigma_X \Lambda_0 ds \\ B^N(\tau_i) = (K_X' + \Lambda_1' \Sigma_X')^{-1} \left\{\exp(-(K_X' + \Lambda_1' \Sigma_X')\tau_i) - \mathrm{I}_{2x2}\right\} \delta_{1,r} \end{cases} \tag{2.11}$$

Since the payout of a bond that matures immediately should equal the face value of 1, we know that $A^N(0) = 0$ and $B^N = 0_{2\times 1}$. As $X_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}$', we have $P^N_{\tau_i}(0, X_0) = \exp\left(A^N(\tau_i) + B^N(\tau_i)' X_0\right) = \exp(A^N(\tau_i))$.

Finally, the original market contains an inflation-linked bond in which the agent cannot invest. We assume that the dynamics of this bond are fixed in the original market, whereas the price of risk of the untraded Brownian motion is modified in the dual optimization problem via the perturbation term. To derive the dynamics of an inflation-linked bond maturing at time τ, we can follow a similar procedure as for the nominal bonds. Since the instantaneous real short rate is affine in the state variables, we can utilize the results of Duffie and Kan (1996) to find that the *real* price of the inflation-indexed bond is given by $P^R_{t+\tau}(t,X_t) = \mathbb{E}\left[\frac{\phi^R_{t+\tau}}{\phi^R_t}\middle|\mathcal{F}_t\right] = \exp(A^R(\tau) + B^R(\tau)'X_t)$. Again, the martingale property of $\phi^R_t P^R_{t+\tau}(t,X_t)$ can be exploited to arrive at this expression. However, as the rest of the economy is defined in nominal terms, we also want to express the price of the inflation linked bond in nominal terms. Therefore, we are interested in the dynamics of $\hat{P}^R_{t+\tau}(t,X_t) = P^R_{t+\tau}(t,X_t)\Pi_t$, which expresses the price dynamics of the inflation linked bond in nominal terms. We find the following dynamics for $\hat{P}^R_{t+\tau}(t,X_t)$ (see Appendix A.1 for derivations):

$$\frac{d\hat{P}^R_{t+\tau}(t,X_t)}{\hat{P}^R_{t+\tau}(t,X_t)} = \Big(r_t + B^R(\tau)'\Sigma_X\Lambda_t + \sigma'_\Pi\Lambda_t\Big)\,dt + \Big(B^R(\tau)'\Sigma_X + \sigma'_\Pi\Big)\,dZ_t \quad (2.12)$$

The functions $A^R(\tau) \in \mathbb{R}$ and $B^R(\tau) \in \mathbb{R}^{2\times 1}$ are specified as follows:

$$\begin{cases} A^R(\tau) = \int_0^\tau -R_{0,R} + \frac{1}{2}B^R(s)'\Sigma_X\Sigma'_X B^R(s) - B^R(s)'\Sigma_X(\Lambda_0 - \sigma_\Pi)ds \\ B^R(\tau) = (K'_X + \Lambda'_1\Sigma'_X)^{-1}\left\{\exp(-(K'_X + \Lambda'_1\Sigma'_X)\tau) - \mathrm{I}_{2x2}\right\}R_{1,R} \end{cases} \quad (2.13)$$

Similar as for the nominal bonds, we have $A^R(0) = 0$, $B^R = 0_{2\times 1}$, and $P^R_\tau(0,X_0) = \exp\Big(A^R(\tau) + B^R(\tau)'X_0\Big) = \exp(A^R(\tau))$.

To conclude this paragraph, we combine the dynamics of all assets in the vector Y_t, inspired by the work Kamma and Pelsser (2022). This integrated representation of the nominal financial market will support further derivations. The vector $Y_t \in \mathbb{R}^{4x1}$ then reads as follows:

$$Y_t = \begin{bmatrix} P^N_{t+\tau_1}(t,X_t) \\ P^N_{t+\tau_2}(t,X_t) \\ \hat{P}^R_{t+\tau}(t,X_t) \\ S_t \end{bmatrix} \quad (2.14)$$

The dynamics of Y_t are specified as:

$$dY_t = \mathrm{diag}(Y_t)\Big((r_t + \Sigma\Lambda_t)dt + \Sigma dZ_t\Big) \quad (2.15)$$

where $\Sigma \in \mathbb{R}^{4\times 4}$ represents the variance-covariance matrix of the financial market:

$$\Sigma = \begin{bmatrix} B^N(\tau_1)'\Sigma_X \\ B^N(\tau_2)'\Sigma_X \\ B^R(\tau)'\Sigma_X + \sigma'_\Pi \\ \sigma'_S \end{bmatrix} = \begin{bmatrix} B^N_1(\tau_1) & B^N_2(\tau_1) & 0 & 0 \\ B^N_1(\tau_2) & B^N_2(\tau_2) & 0 & 0 \\ B^R_1(\tau) + \sigma_{\Pi,1} & B^R_2(\tau) + \sigma_{\Pi,2} & \sigma_{\Pi,3} & \sigma_{\Pi,4} \\ \sigma_{S,1} & \sigma_{S,2} & \sigma_{S,3} & \sigma_{S,4} \end{bmatrix} \quad (2.16)$$

2.2 Portfolio optimization problem

In this section, we solve the portfolio optimization problem in the KNW model. In the original incomplete market, the agent faces the following optimization problem:

$$\sup_{\theta} \mathbb{E}\Big[\frac{(W_T/\Pi_T)^{1-\gamma}}{1-\gamma}\Big] \\ \text{s.t. } dW_t = W_t(r_t + \theta_t'\Sigma\Lambda_t)dt + W_t\theta_t'\Sigma dZ_t \tag{2.17}$$

where $\theta_t \in \mathbb{R}^{4\times1}$ represents an investment strategy in the original incomplete market. Note that the agent is interested in optimizing real terminal wealth, whereas investments can only be made in nominal assets. $\theta_{3,t}$ should be zero at each time point due to the trading constraint. Therefore, the static optimization problem corresponding to (2.17) is characterized by infinitely many pricing kernels and thus the Martingale method cannot be applied. Therefore, we find the optimal investment strategy via duality theory. The asset dynamics in the dual market are as given in the primal market. The only differences between the two markets are (i) the lifted trading constraint on the inflation-indexed bond in the dual market, leading to a complete market and (ii) the use of a perturbation term so that we can make the dual strategy admissible in the primal market. To satisfy the trading constraint in the primal market we thus let the price of risk of the untraded Brownian motion be influenced by a perturbation term in the dual market. As a technical consequence of the duality theory, the price of risk in the dual market of the Brownian motion untraded in the original market equals the price of risk in the original market plus a perturbation term. In line with the duality theory proposed by Karatzas et al. (1991), Cvitanic and Karatzas (1992), Xu and Shreve (1992), and Kamma (2023), we therefore postulate that the price of risk in the optimization problem is given as follows:

$$\begin{aligned}\tilde{\Lambda}_t &= \Lambda_t + a(X_t)e_3 \\ &= \tilde{\Lambda}_0 + \tilde{\Lambda}_1 X_t\end{aligned} \tag{2.18}$$

where Λ_t is the price of risk corresponding to the original market defined in (2.6), $a(X_t)$ the perturbation term, and $e_3 = \begin{bmatrix}0\ 0\ 1\ 0\end{bmatrix}'$. After we have found the optimal strategy in the dual market, we determine $a(X_t)$ such that no investment in the inflation-indexed bond is made. Note that we implicitly assume that $a(X_t)$ is a linear function of the state variables, so that we can determine the vectors $\tilde{\Lambda}_0$ and $\tilde{\Lambda}_1$[11]. Hence, in the KNW model, choosing the perturbation term corresponds to choosing a constant and linear coefficient that describes the affine structure of the price of risk in the dual market. Furthermore, the demand of the inflation-linked bond in the dual market only depends on the price of risk and state variables at time t. Therefore, it suffices to assume that the perturbation term only depends on X_t and not on its past values.

The modification of the price of risk of the untraded Brownian motion can

[11] Without the assumption that the perturbation term is affine in the state variables, it is in general not possible to make the dual strategy admissible in the primal market and thus we impose this form ex-ante.

intuitively be explained by the fact that the agent is not allowed to invest in the inflation-indexed bond in the original market. Therefore, the agent is not interested in this price of risk and thus we can modify the price of risk in the dual market via the perturbation term. From a mathematical perspective the choice of $a(X_t)$ that reduces the demand of the inflation-linked bond to zero corresponds to minimizing the dual objective, which is equivalent to solving the primal optimization problem. Using $a(X_t)$, we can thus find the optimal investment strategy in the original incomplete market by solving the dual optimization problem. The primal market is thus equal to the dual market up to the specification of $a(X_t)$. Consequently, $a(X_t)$ outlines the market price of risk corresponding to the unhedgeable sources of risk.

As a consequence of working with a modified price of risk in the dual market, the dynamics of the nominal and real pricing kernel change. Although the dynamics of the nominal pricing kernel in the dual market trivially follow from the modified price of risk, we choose to present it to ensure clarity:

$$\frac{d\tilde{\phi}_t^N}{\tilde{\phi}_t^N} = -r_t dt - \tilde{\Lambda}_t' dZ_t \tag{2.19}$$

where $\tilde{\phi}_0^N = 1$. Similarly to the primal market, we define the real pricing kernel in the dual market as $\tilde{\phi}_t^R = \tilde{\phi}_t^N \Pi_t$.Using the derivations of the real pricing kernel in the primal market from Appendix A.1, we find the following dynamics in the dual market:

$$\begin{aligned}\frac{d\tilde{\phi}_t^R}{\tilde{\phi}_t^R} &= -(r_t - \pi_t + \sigma_\Pi' \tilde{\Lambda}_t) - (\tilde{\Lambda}_t - \sigma_\Pi')dZ_t \\ &= -\tilde{R}_t dt - \tilde{\Lambda}_t^{R\prime} dZ_t\end{aligned} \tag{2.20}$$

where $\tilde{\Lambda}_t^R$ is the real pricing kernel in the dual market and $\tilde{\phi}_0^R = 1$. $\tilde{R}_t$ can be interpreted as a modified real instantaneous interest rate, specific to the dual optimization problem. In nominal terms, the change from the primal to the dual problem is characterized by the choice of the perturbation term that reduces the demand of the inflation-linked bond to zero. However, we are optimizing the agent's real wealth and thus the choice of the perturbation term works through in the real market; it does so in the modified real interest rate above. The difference between the instantaneous real interest rate in the dual market and primal market can thus be interpreted as a technical consequence from performing the optimization problem in real terms rather than in nominal terms. An intuitive explanation for $\tilde{R}_t$ could be that it is the correction of the instantaneous interest rate R_t in the dual market for the fact that the agent cannot hedge inflation risk in the primal market. To finalize the introduction of the real pricing kernel in the dual market, we present the specification of

$\tilde{R}_t$:

$$\begin{aligned}\tilde{R}_t &= r_t - \pi_t + \sigma_\Pi' \tilde{\Lambda}_t \\ &= (\delta_{0,r} - \delta_{0,\pi} + \sigma_\Pi' \tilde{\Lambda}_0) + (\delta_{1,r} - \delta_{1,\pi} + \tilde{\Lambda}_1' \sigma_\Pi)' X_t \\ &= \tilde{R}_{0,R} + \tilde{R}_{1,R}' X_t\end{aligned} \tag{2.21}$$

Based on the price of risk particular to the dual optimization problem in (2.18), we can find the budget constraint in the dual market. The agent's total *nominal dual* wealth evolves according to the following SDE:

$$d\tilde{W}_t = \tilde{W}_t(r_t + \tilde{\theta}_t(a)'\Sigma\tilde{\Lambda}_t)dt + \tilde{W}_t\tilde{\theta}_t(a)'\Sigma dZ_t \tag{2.22}$$

where r_t is the nominal short rate specified in (2.2), $\tilde{\Lambda}_t$ the price of risk in the dual optimization problem specified in (2.18), and Σ the variance-covariance matrix from the original market specified in (2.16). $\tilde{\theta}_t(a) \in \mathbb{R}^{4\times 1}$ describes an investment strategy in the dual market. Note that we explicitly stress the dependence of this strategy on the perturbation term $a(X_t)$. Consequently, after we have found the optimal perturbation term we choose to describe the optimal investment strategy in the primal market as $\theta_t(a^*) \in \mathbb{R}^{4\times 1}$. We thus remove the tilde to stress that the vector is specified in the primal market. Furthermore, we stress that the optimal strategy in the original market is dependent on the optimal perturbation term. Each element of $\theta_t(a^*)$ corresponds to the fraction of total primal wealth W_t that is invested in the corresponding asset. For example, the first element of $\theta_t(a^*)$ corresponds to the position in the original market in the nominal bond with maturity τ_1 whereas the fourth element shows the position in the stock. The remainder $1 - \sum_{i=1}^{4} \theta_{i,t}(a^*)$ is invested in the money market account. The budget constraint states that the agent earns the risk premium $\Sigma\tilde{\Lambda}_t$ on the fraction of wealth that is invested in the risky assets. The remainder is invested in the money market account, paying the nominal short rate r_t. As soon as we have found $a(X_t)$ that makes the optimal dual strategy admissible in the primal market, we can define the optimal *nominal primal* wealth. In fact, after inserting the dual-optimal perturbation term in the optimal dual wealth, this is exactly equal to the optimal primal wealth (Cvitanic and Karatzas, 1992). Finding optima for the primal problem can thus completely be restricted to solving the dual-problem. We will denote primal wealth as W_t, i.e. without the tilde which is used for dual wealth.

By choosing $\tilde{\theta}_t(a)$ optimally, the agent aims at maximizing the expected utility received from real terminal dual wealth. Note that the agent can only invest in nominal assets. This brings out the dynamic optimization problem the agent faces:

$$\begin{aligned}&\sup_\theta \; \mathbb{E}\Big[\frac{(\tilde{W}_T/\Pi_T)^{1-\gamma}}{1-\gamma}\Big] \\ &\text{s.t. } d\tilde{W}_t = \tilde{W}_t(r_t + \tilde{\theta}_t(a)'\Sigma\tilde{\Lambda}_t)dt + \tilde{W}_t\tilde{\theta}_t(a)'\Sigma dZ_t\end{aligned} \tag{2.23}$$

Since the trading constraint on the inflation-linked bond is lifted in the dual market, the agent acts in a complete market. We can therefore view wealth as a traded asset and consequently transfer the dynamic problem above to a static optimization problem, following the logic of Karatzas et al. (1987) and Cox and Huang (1989, 1991). Consequently, both the budget constraint and control variable change. In the static optimization problem we therefore optimize terminal wealth, leading to the following problem:

$$\begin{aligned} &\sup_{\tilde{W}_T} \mathbb{E}\Big[\frac{(\tilde{W}_T/\Pi_T)^{1-\gamma}}{1-\gamma}\Big] \\ &\text{s.t. } \mathbb{E}\Big[\tilde{\phi}_T^N \tilde{W}_T\Big] = W_0 \end{aligned} \tag{2.24}$$

We resolve to a Lagrangian approach when solving the static optimization problem above. We find the following Lagrangian $\mathcal{L}$, for some Lagrange multiplier l:

$$\begin{aligned} \mathcal{L} &= \mathbb{E}\Big[\frac{(\tilde{W}_T/\Pi_T)^{1-\gamma}}{1-\gamma}\Big] + l\Big(W_0 - \mathbb{E}\Big[\tilde{\phi}_T^N \tilde{W}_T\Big]\Big) \\ &= \mathbb{E}\Big[\frac{(\tilde{W}_T/\Pi_T)^{1-\gamma}}{1-\gamma} - l\tilde{\phi}_T^N \tilde{W}_T\Big] + lW_0 \end{aligned} \tag{2.25}$$

Taking derivatives inside the expectation leads to the following FOC:

$$\frac{\tilde{W}_T^{*-\gamma}}{\Pi_T^{1-\gamma}} - l\tilde{\phi}_T^N = 0 \tag{2.26}$$

Rewriting gives the following expression for optimal terminal wealth:

$$\tilde{W}_T^* = (l\tilde{\phi}_T^N \Pi_T^{1-\gamma})^{-\frac{1}{\gamma}} \tag{2.27}$$

Plugging the result above in the static budget constraint from (2.24) gives us an expression for the Lagrange multiplier l:

$$l = W_0^{-\gamma}\left(\mathbb{E}\Big[(\tilde{\phi}_T^N \Pi_T)^{1-\frac{1}{\gamma}}\Big]\right)^{\gamma} = W_0^{-\gamma} g_T^{\gamma} \tag{2.28}$$

where we thus have defined g_T as $\mathbb{E}\Big[(\tilde{\phi}_T^N \Pi_T)^{1-\frac{1}{\gamma}}\Big]$. Substituting the Lagrange multiplier back in (2.27) leads to the following expression for optimal terminal dual wealth:

$$\tilde{W}_T^* = \frac{W_0}{g_T}(\tilde{\phi}_T^N \Pi_T^{1-\gamma})^{-\frac{1}{\gamma}} \tag{2.29}$$

By using the Martingale method, we treat wealth as a traded asset and thus we can find the optimal dual wealth at time t by discounting the optimal terminal dual wealth with the appropriate pricing kernel (Schumacher, 2020):

$$\begin{aligned}
\tilde{W}_t^* &= \frac{1}{\tilde{\phi}_t^N}\mathbb{E}\left[\tilde{\phi}_T^N \tilde{W}_T^* | \mathcal{F}_t\right] \\
&= \frac{1}{\tilde{\phi}_t^N}\frac{W_0}{g_T}\mathbb{E}\Big[(\tilde{\phi}_T^N \Pi_T)^{1-\frac{1}{\gamma}}\Big|\mathcal{F}_t\Big] \\
&= \frac{W_0}{g_T}(\tilde{\phi}_t^N)^{-\frac{1}{\gamma}}\Pi_t^{1-\frac{1}{\gamma}}\mathbb{E}\left[(\frac{\tilde{\phi}_T^N \Pi_T}{\tilde{\phi}_t^N \Pi_t})^{1-\frac{1}{\gamma}}|\mathcal{F}_t\right] \\
&= \frac{W_0}{g_T}(\tilde{\phi}_t^N)^{-\frac{1}{\gamma}}\Pi_t^{\frac{\gamma-1}{\gamma}}\mathbb{E}\left[(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}|\mathcal{F}_t\right]
\end{aligned} \tag{2.30}$$

To facilitate further derivations, we focus on finding an explicit expression for $\mathbb{E}\Big[(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}\Big|\mathcal{F}_t\Big]$. In Appendix A.2 we show that $(\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}}$ is an exponentially affine quadratic function of the state variables. Therefore we can solve the conditional expectation in our optimization problem by the results of Duffie and Kan (1996) on affine yield models, similarly to how we determined bond prices in Section 2.1[12]. By utilizing the affine state variable structure, we find the following structure for the conditional expectation:

$$\tilde{P}(t, X_t) = \mathbb{E}\left[(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}\Big|\mathcal{F}_t\right] = \exp\Big(\tilde{A}(t) + \tilde{B}(t)'X_t + X_t'\tilde{C}(t)X_t\Big) \tag{2.31}$$

where $\tilde{A}(t) \in \mathbb{R}$, $\tilde{B}(t) \in \mathbb{R}^{2\times1}$, and $\tilde{C}(t) \in \mathbb{R}^{2\times2}$. We find the following system of ODE's that describes $\tilde{A}(t)$, $\tilde{B}(t)$, and $\tilde{C}(t)$:

$$\begin{aligned}
\dot{\tilde{A}}(t) &= \frac{\gamma-1}{\gamma}\tilde{B}(t)'\Sigma_X(\tilde{\Lambda}_0 - \sigma_\Pi) - \frac{1}{2}\tilde{B}(t)'\tilde{B}(t) - \text{tr}(\tilde{C}(t)) + \frac{\gamma-1}{\gamma}\tilde{R}_{0,R} \\
&\quad + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\tilde{\Lambda}_0 - \sigma_\Pi)'(\tilde{\Lambda}_0 - \sigma_\Pi) \\
\dot{\tilde{B}}(t) &= \left(\frac{\gamma-1}{\gamma}\tilde{\Lambda}_1'\Sigma_X' + K_X' - 2\tilde{C}(t)'\right)\tilde{B}(t) + 2\frac{\gamma-1}{\gamma}\tilde{C}(t)'\Sigma_X(\tilde{\Lambda}_0 - \sigma_\Pi) \\
&\quad + \frac{\gamma-1}{\gamma}\tilde{R}_{1,R} + \frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_1'(\tilde{\Lambda}_0 - \sigma_\Pi) \\
\dot{\tilde{C}}(t) &= 2\left(K_X' + \frac{\gamma-1}{\gamma}\tilde{\Lambda}_1'\Sigma_X'\right)\tilde{C}(t) - 2\tilde{C}(t)'\tilde{C}(t) + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_1'\tilde{\Lambda}_1
\end{aligned} \tag{2.32}$$

By combining (2.29) and (2.30), we know that $\mathbb{E}\left[(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_T^R})^{\frac{\gamma-1}{\gamma}}|\mathcal{F}_t\right] = 1$ and thus we find the terminal conditions $\tilde{A}(T) = 0$, $\tilde{B}(T) = 0_{2\times1}$, and $\tilde{C}(T) = 0_{2\times2}$. For derivations of (2.31) and (2.32), we refer to Appendix A.2. The system of ODE's that describes $\tilde{A}(t)$, $\tilde{B}(t)$, and $\tilde{C}(t)$ involves a Riccati equation in

[12] The conditional expectation we are interested in here takes a form similar to the conditional expectation that determines bond prices, see Appendix A.1 in combination with Appendix A.2. We therefore find a similar structure, with the only difference the expectation here is affine quadratic in the state variables rather than affine.

matrix form. A Riccati equation is a quadratic first order differential equation (Polyanin and Zaitsev, 2002). As such, we know that we cannot find an analytical solution for the system (Kamma and Pelsser, 2022). Hence, for any implementation of the system above we have to result to numerical approximations. Note that the system (partly) describes the evolution of optimal dual wealth. Hence, it is dependent on the perturbation term $a(X_t)$. Once the choice of $a(X_t)$ that reduces the demand of the inflation-linked bond to zero is made, we have found the system that corresponds to the primal market.

By applying the Martingale method, we treat wealth as a traded asset. Therefore we know that the optimal nominal dual wealth, discounted with the nominal pricing kernel, should be a martingale. First, note that from (2.30) we obtain the following expression for $\tilde{W}_t^* \tilde{\phi}_t^N$:

$$\begin{aligned} \tilde{W}_t^* \tilde{\phi}_t^N &= \frac{W_0}{g_T} (\tilde{\phi}_t^N)^{-\frac{1}{\gamma}} \Pi_t^{\frac{\gamma-1}{\gamma}} \tilde{P}(t, X_t) \tilde{\phi}_t^N \\ &= \frac{W_0}{g_T} (\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}} \tilde{P}(t, X_t) \end{aligned} \tag{2.33}$$

In Appendix A.2 we find that $\tilde{W}_t^* \tilde{\phi}_t^N$ adheres to the following dynamics:

$$d\tilde{W}_t^* \tilde{\phi}_t^N = \tilde{W}_t^* \tilde{\phi}_t^N \left((\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma_X - \frac{\gamma - 1}{\gamma} \tilde{\Lambda}_t^{R\prime} \right) dZ_t \tag{2.34}$$

Note that the budget constraint in (2.22) provides a different formulation of the evolution of optimal nominal dual wealth. Hence, discounting this formulation of the optimal wealth dynamics with the nominal pricing kernel should also lead to a martingale. We therefore find the following alternative formulation of $\tilde{W}_t^* \tilde{\phi}_t^N$ in Appendix A.2:

$$d\tilde{W}_t^* \tilde{\phi}_t^N = \tilde{W}_t^* \tilde{\phi}_t^N (\tilde{\theta}_t^*(a)' \Sigma - \tilde{\Lambda}_t') dZ_t \tag{2.35}$$

Note that at this point we have not yet determined the optimal perturbation term and thus our optimal strategy in the dual market depends on $a(X_t)$ rather than $a^*(X_t)$.

Equating the volatility terms of the two different formulations of discounted optimal wealth in (2.34) and (2.35) leads to an equation from which the optimal investment strategy in the dual market can be retrieved:

$$(\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma_X - \frac{\gamma - 1}{\gamma} \tilde{\Lambda}_t^{R\prime} = \tilde{\theta}_t^*(a)' \Sigma - \tilde{\Lambda}_t' \tag{2.36}$$

By solving the equation above for $\tilde{\theta}_t^*(a)'$, we find the following optimal investment strategy in the dual market (further derivations can be found in

Appendix A.2):

$$\tilde{\theta}_t^*(a)' = \frac{1}{\gamma}\left(\Sigma^{-1}\right)^\top \tilde{\Lambda}_t + \frac{\gamma-1}{\gamma}\left(\Sigma^{-1}\right)^\top \sigma_\Pi + \left(\Sigma^{-1}\right)^\top \tilde{H}(t, X_t) \tag{2.37}$$

Note that this strategy is still dependent on $a(X_t)$ and thus it is not yet admissible in the primal market. $\tilde{H}(t, X_t)$ is a time dependent function of the state variables:

$$\tilde{H}(t, X_t) = \begin{bmatrix} \tilde{B}_{1,t} + 2\tilde{C}_{11,t}X_{1,t} + 2\tilde{C}_{12,t}X_{2,t} \\ \tilde{B}_{2,t} + 2\tilde{C}_{21,t}X_{1,t} + 2\tilde{C}_{22,t}X_{2,t} \\ 0 \\ 0 \end{bmatrix} \tag{2.38}$$

Now that we have found the optimal strategy in the dual market it remains to be answered how we can link the solution to a strategy in the original incomplete market. From (2.37) we see that in general part of the agent's wealth will be allocated to the inflation-indexed bond, i.e. $\tilde{\theta}_{3,t}^* \neq 0$. Therefore we have to determine the perturbation term $a(X_t)$ in the dual market that reduces the demand of the inflation-indexed bond to zero. This choice of $a(X_t)$ corresponds to minimizing the dual objective and thereby maximizing the primal objective. Before finding the optimal perturbation term, note that the variance-covariance matrix defined in (2.16) contains a two by two block matrix with zeroes. Although it is technically demanding to find the inverse of the variance covariance matrix analytically, we can utilize the block matrix structure. We therefore know that Σ^{-1} has the following form:

$$\Sigma^{-1} = \begin{bmatrix} (\Sigma^{-1})_{11} & (\Sigma^{-1})_{12} & 0 & 0 \\ (\Sigma^{-1})_{21} & (\Sigma^{-1})_{22} & 0 & 0 \\ (\Sigma^{-1})_{31} & (\Sigma^{-1})_{32} & (\Sigma^{-1})_{33} & (\Sigma^{-1})_{34} \\ (\Sigma^{-1})_{41} & (\Sigma^{-1})_{42} & (\Sigma^{-1})_{43} & (\Sigma^{-1})_{44} \end{bmatrix} \tag{2.39}$$

Note that we write $(\Sigma^{-1})_{ij}$ for element i, j of the inverse of Σ, which is unequal to $(\Sigma_{ij})^{-1}$. By using the inverse variance covariance matrix, it is found that the optimal demand for the inflation-linked bond in the dual market at time t looks as follows:

$$\begin{aligned} \tilde{\theta}_{3,t}^*(a) = &\left(\tilde{H}_1(t, X_t) + \frac{1}{\gamma}(\Lambda_{0,1} + \Lambda_{1,(1,1)}X_{1,t} + \Lambda_{1,(1,2)}X_{2,t}) + \frac{\gamma-1}{\gamma}\sigma_{\Pi,1}\right) \cdot 0 \\ &+ \left(\tilde{H}_2(t, X_t) + \frac{1}{\gamma}(\Lambda_{0,2} + \Lambda_{1,(2,1)}X_{1,t} + \Lambda_{1,(2,2)}X_{2,t}) + \frac{\gamma-1}{\gamma}\sigma_{\Pi,2}\right) \cdot 0 \\ &+ \left(\frac{1}{\gamma}(\Lambda_{0,3} + \Lambda_{1,(3,1)}X_{1,t} + \Lambda_{1,(3,2)}X_{2,t} + a(X_t)) + \frac{\gamma-1}{\gamma}\sigma_{\Pi,3}\right) \cdot (\Sigma^{-1})_{33} \\ &+ \left(\frac{1}{\gamma}(\Lambda_{0,4} + \Lambda_{1,(4,1)}X_{1,t} + \Lambda_{1,(4,2)}X_{2,t}) + \frac{\gamma-1}{\gamma}\sigma_{\Pi,4}\right) \cdot (\Sigma^{-1})_{43} \end{aligned} \tag{2.40}$$

Hence, the demand of the inflation linked bond will equal zero at time t for the following optimal choice of the perturbation term:

$$\begin{aligned} a^*(X_t) &= -(\Lambda_{0,3} + \Lambda_{1,(3,1)}X_{1,t} + \Lambda_{1,(3,2)}X_{2,t}) - (\gamma-1)\sigma_{\Pi,3} \\ &\quad - (\Lambda_{0,4} + \Lambda_{1,(4,1)}X_{1,t} + \Lambda_{1,(4,2)}X_{2,t} + (\gamma-1)\sigma_{\Pi,4})(\Sigma^{-1})_{43}((\Sigma^{-1})_{33})^{-1} \\ &= -\Lambda_{3,t} - (\gamma-1)\sigma_{\Pi,3} - (\Lambda_{4,t} + (\gamma-1)\sigma_{\Pi,4})(\Sigma^{-1})_{43}((\Sigma^{-1})_{33})^{-1} \end{aligned} \tag{2.41}$$

Note that the optimal perturbation term depends on the risk aversion parameter γ, therefore the perturbation term is agent specific. The choice of $a^*(X_t)$ above links the solution to the dual optimization problem to the solution of the primal optimization problem. This choice of $a^*(X_t)$ pins down the constant and linear coefficient of the price of risk in the dual market, as shown in (2.18). Consequently, we have determined all parameters such that system of ODE's in (2.32) can be linked to the original market. Formally, the vectors $\tilde{\Lambda}_0$ and $\tilde{\Lambda}_1$ that make the dual strategy admissible in the primal market are then defined below:

$$\begin{aligned} \tilde{\Lambda}_t &= \Lambda_t + a^*(X_t)e_3 \\ &= \Lambda_0 - \Big(\Lambda_{0,3} + (\gamma-1)\sigma_{\Pi,3} + (\Lambda_{0,4} + (\gamma-1)\sigma_{\Pi,4})(\Sigma^{-1})_{43}((\Sigma^{-1})_{33})^{-1}\Big)e_3 + \Big(\Lambda_1 \\ &\quad - e_3\Big[\Lambda_{1,(3,1)} + \Lambda_{1,(4,1)}(\Sigma^{-1})_{43}((\Sigma^{-1})_{33})^{-1}\ \Lambda_{1,(3,2)} + \Lambda_{1,(4,2)}(\Sigma^{-1})_{43}((\Sigma^{-1})_{33})^{-1}\Big]\Big)X_t \\ &= \tilde{\Lambda}_0 + \tilde{\Lambda}_1 X_t \end{aligned} \tag{2.42}$$

Combining all information leads to the optimal investment strategy in the original incomplete market, which we formally introduce in Proposition 2.1:

Proposition 2.1 *Consider the optimal dynamic investment problem in the KNW model specified in* (2.17)*. The corresponding optimal wealth process in the incomplete market, W_t^*, is given by:*

$$W_t^* = \frac{W_0}{g_T}(\tilde{\phi}_t^N)^{-\frac{1}{\gamma}}\Pi_t^{\frac{\gamma-1}{\gamma}}\tilde{P}(t, X_t)$$

for all $t \in [0, T]$. Here, $g_T = \mathbb{E}\Big[(\tilde{\phi}_T^N\Pi_T)^{1-\frac{1}{\gamma}}\Big]$, $\tilde{\phi}_t^N$ is defined in (2.19)*, and $\tilde{P}(t, X_t) = \exp\Big(\tilde{A}(t) + \tilde{B}(t)'X_t + X_t'\tilde{C}(t)X_t\Big)$. The ODE's that jointly determine $\tilde{A}(t)$, $\tilde{B}(t)$, and $\tilde{C}(t)$ are given in* (2.32)*. $\tilde{\phi}_t^N$ and $\tilde{P}(t, X_t)$ depend on the optimal perturbation term $a^*(X_t)$, which is set such that an agent does not invest in the inflation-linked bond:*

$$a^*(X_t) = -\Lambda_{3,t} - (\gamma-1)\sigma_{\Pi,3} - (\Lambda_{4,t} + (\gamma-1)\sigma_{\Pi,4})(\Sigma^{-1})_{43}((\Sigma^{-1})_{33})^{-1}$$

Ultimately, this leads to the optimal investment strategy in the incomplete KNW market:

$$\theta_t^*(a^*) = \frac{1}{\gamma}\left(\Sigma^{-1}\right)^\top \tilde{\Lambda}_t + \frac{\gamma-1}{\gamma}\left(\Sigma^{-1}\right)^\top \sigma_\Pi + \left(\Sigma^{-1}\right)^\top \tilde{H}(t, X_t)$$

where Σ, $\tilde{H}(t, X_t)$, *and* $\tilde{\Lambda}_t$ *are defined in* (2.16), (2.38), *and* (2.42), *respectively. Both* $\tilde{\Lambda}_t$ *and* $\tilde{H}(t, X_t)$ *depend on the choice of* $a^*(X_t)$ *above. The remainder* $1 - \sum_{i=1}^{4} \theta^*_{i,t}(a^*)$ *is invested in the money market account.*

First of all, it should be noted that all quantities in Proposition 2.1 defined with a tilde depend on the price of risk of the Brownian motion untraded in the original market. Hence, their definition in the dual market is characterized by the, then unknown, perturbation term $a(X_t)$. To make the optimal dual strategy admissible in the primal market, the definitions of those terms in the primal market have to be modified slightly. This modification is driven by the choice of $a(X_t)$ in (2.41), which reduces the demand of the inflation-linked bond to zero.

In line with three-fund separation introduced by Merton (1973), we see that the optimal demand in the incomplete market consists of three terms: mean-variance optimal demand and a combination of two hedge demands. The mean variance demand is defined, similar to Merton, as the price of risk divided by the product of the volatility and the risk aversion parameter. In other words, the mean-variance demand is the portfolio that receives the highest Sharpe ratio (Sangvinatsos and Wachter, 2005). The first hedge demand depends on the volatility of the CPI (σ_Π). Mathematically speaking, the first hedge demand can be seen as the projection of the volatility of the CPI on the asset returns. Therefore it can be interpreted as an adjustment for the fact that the agent can only invest in nominal securities, whereas we are optimizing real wealth. In other words, this term creates demand to hedge against realized inflation. Note that by the trading constraints the agent cannot use the inflation-indexed bond as instrument to hedge against realized inflation. However, depending on the values of σ_Π and Σ, the other assets might also be used to (partly) hedge against the realized inflation[13]. Since the values of σ_Π and Σ are the same for each agent, the first hedge demand will lead to the same allocations for each agent, independent of the risk aversion parameter. The second hedge demand depends on the vector $\tilde{H}(t, X_t)$. This vector is the product of the sensitivity of $\tilde{P}(t, X_t)$ with respect to the state variables, and the variance of the state variables. Hence, $\tilde{H}(t, X_t)$ measures the sensitivity of the evolution of optimal wealth with respect to the state variables. From a mathematical line of thought the second hedge demand can be seen as the projection of asset returns on the state variables. The last two elements of $\tilde{H}(t, X_t)$ are zero and thus only the nominal bonds will be used to hedge against fluctuations in the state variables. This should come as no surprise since, given the structure of the model, the purpose of the state variables is to facilitate the randomness in the term structure. Since $\tilde{H}(t, X_t)$ is dependent on the agent specific perturbation term $a(X_t)$, the degree to which the nominal bonds will be used to hedge against fluctuations in the state variables will be different across agents.

[13]Note that we assume $\sigma_{\Pi,4} = 0$. Therefore only the two nominal bonds will be used to hedge against realized inflation, in contrast to the stock.

We conclude this paragraph by comparing the optimal investment strategy defined in Proposition 2.1 to the optimal strategies found in the financial market models of Brennan and Xia (2002) and Sangvinatsos and Wachter (2005). In line with the findings in this thesis, both papers find that optimal demand consists of three funds, where the first fund is the mean variance optimal demand. Both Brennan and Xia (2002) and Sangvinatsos and Wachter (2005) find the two hedge demands to be dependent on realized inflation and the state variables. Hence, we conceptually find a similar strategy, however, due to subtle differences in model setup the actual composition of the hedge demands will differ per model. Furthermore, we have used duality theory to solve the portfolio optimization problem. Both Brennan and Xia (2002) and Sangvinatsos and Wachter (2005) find the optimal strategy by choosing the particular pricing kernel that reduces the demand of the untraded asset to zero. This forces ex-post interruptions in the financial market. We have proposed a method to solve the portfolio optimization problem in a similar model, without the need for ex-post modifications of the market. We thus extend the literature by mathematically formalizing the results of Brennan and Xia (2002) and Sangvinatsos and Wachter (2005).

2.3 Numerical implementation investment strategy

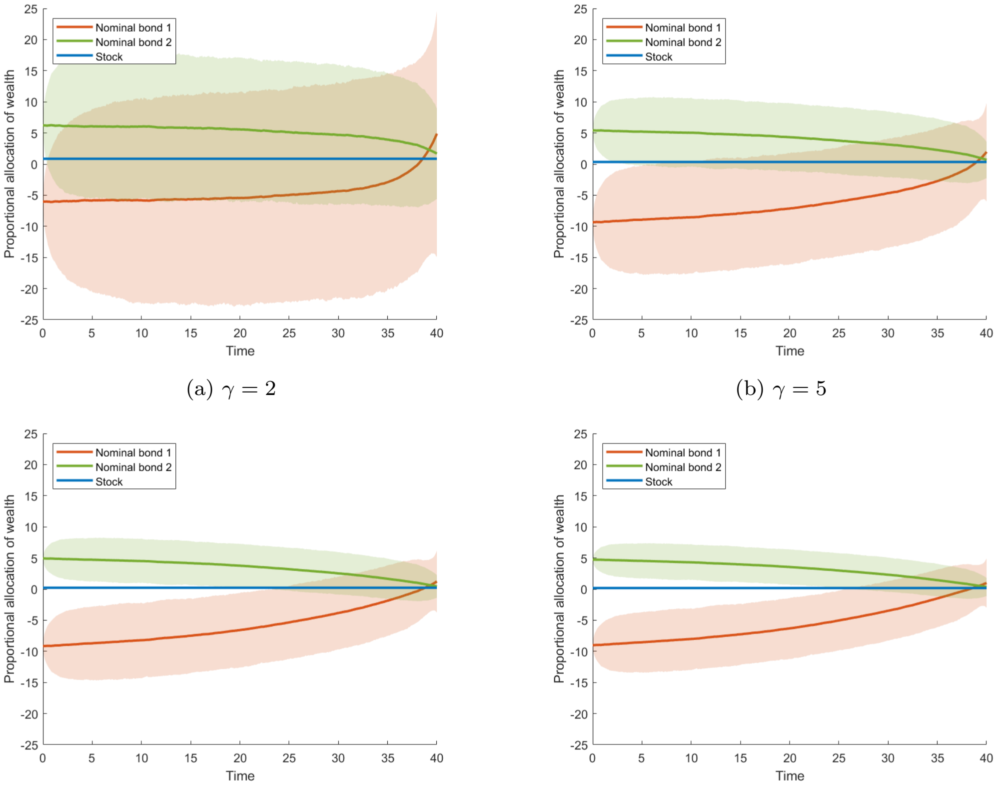
In this section, we present results of a numerical implementation of the investment strategy. As baseline input, we take the calibrated DNB parameter set from the paper of Muns (2015)[14]. These estimates have been used by the DNB to create the scenarioset in the second quarter of 2015 and are based on the calibrated estimates of Draper (2012) (Muns, 2015). The parameter values are presented in Table 2.1. We assume an investment horizon of $T = 40$ years, with monthly time steps used in the simulations. We assume the two nominal bonds to have a time to maturity of $\tau_1 = 1$ and $\tau_2 = 5$, respectively. Note that this implies that the agent can invest in a nominal bond with maturity 1 or 5 years at each point in time. The agent cannot invest in the inflation-linked bond due to the existence of trading constraints. We assume that the inflation-linked bond has a time to maturity equal to $\tau = 5$. We simulate $m = 10,000$ scenarios with initial wealth $W_0 = 1$.

[14]The estimates can be found in the paper of Muns (2015) in Table 2, column 3.

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
S_t		Λ_t		Π_t, X_t		r_t, π_t	
η_S	4.52 %	$\Lambda_{0,1}$	0.280	$\sigma_{\Pi,1}$	0.02%	$\delta_{0,r}$	2.40%
$\sigma_{S,1}$	-0.53%	$\Lambda_{0,2}$	0.027	$\sigma_{\Pi,2}$	$-0.568 \cdot 10^{-4}$	$\delta_{1,r(1)}$	-1.48%
$\sigma_{S,2}$	-0.76%	$\Lambda_{1,(1,1)}$	0.149	$\sigma_{\Pi,3}$	0.61%	$\delta_{1,r(2)}$	0.53%
$\sigma_{S,3}$	-2.11%	$\Lambda_{1,(1,2)}$	-0.381	$K_{X(1,1)}$	7.63%	$\delta_{0,\pi}$	2.00%
$\sigma_{S,4}$	16.59%	$\Lambda_{1,(2,1)}$	0.089	$K_{X(2,1)}$	-19.00%	$\delta_{1,\pi(1)}$	-0.63%
		$\Lambda_{1,(2,2)}$	-0.083	$K_{X(2,2)}$	35.25%	$\delta_{1,\pi(2)}$	0.14%

Table 2.1: Benchmark values for the KNW parameters, based on the calibrated DNB estimates presented in Muns (2015). Similar to Koijen et al. (2010) we define $\Lambda_{0,3} = \Lambda_{1,(3,1)} = \Lambda_{1,(3,2)} = \sigma_{\Pi,4} = 0$. In combination with the estimates for the first two elements of Λ_t and η_S this pins down the value of $\Lambda_{0,4}$, $\Lambda_{1,(4,1)}$, and $\Lambda_{1,(4,2)}$.

We stress that the investment strategy presented in Proposition 2.1 will be simulated, which is dependent on the particular choice of $a(X_t)$ that reduces the demand of the inflation-indexed bond to zero. In Figure 2.1, we show the average allocation to the two nominal bonds and stock over the investment horizon for different values of γ, along with the 5th and 95th percentile allocation. The allocation to the inflation-indexed bond is zero by construction and therefore excluded from the graphs. We have used the same random numbers for all simulations in this chapter. Hence, the only difference between the different sub-plots in Figure 2.1 is the different risk aversion parameter underlying the strategy.



(a) $\gamma = 2$ (b) $\gamma = 5$

(c) $\gamma = 8$ (d) $\gamma = 10$

Figure 2.1: Comparison of the optimal strategy in the KNW model between different values of γ for the baseline parameters. The average allocations to the stock and two nominal bonds over $10,000$ simulations of the optimal investment strategy are shown in thick lines. The 90% percentile range is shown in the shaded areas. We assume $T = 40$ with monthly time steps in the simulations. Furthermore, $\tau_1 = 1$, $\tau_2 = 5$, $\tau = 5$, and $W_0 = 1$. Other parameter values are given in Table 2.1.

First of all, it should be noted that there exists a percentile range around the average stock allocation, although it is not visible in the graphs. For $\gamma = 2$ this interval ranges from approximately $\pm 10\%$ of the average allocation, independent of time. For $\gamma = 10$ the interval shrinks to approximately $\pm 3\%$ of the average allocation. For values of γ between 2 and 10 the interval decreases from 10% to 3%. We thus see that the allocation to the stock is rather deterministic, in contrast to the allocations to the two bonds. Furthermore, the average allocation to the stock decreases as the agent gets more risk averse, aligning with our expectations. From a mathematical perspective the pattern in the allocation to the stock can be explained from the optimal strategy in Proposition 2.1. The optimal strategy consists of three terms, from which only the first term influences the allocation to the stock since $\sigma_{\Pi,4} = 0$ and $\tilde{H}_4(t, X_t) = 0$ for every t. In other words, all randomness in the allocation to the stock comes from the randomness in the fourth element of the price of risk. For the random numbers used to generate Figure 2.1, the average value per time point of the fourth element in the price of risk equals 0.2827, with a standard deviation of 0.02. Since the fourth element of the price of risk, which is the only random source influencing the allocation to the stock, has small variability, we find relatively small variability in the stock allocation.

For the allocations to the two nominal bonds we see a more extreme pattern. To optimally hedge against interest rate risk and inflation risk without taking a position in the inflation-linked bond, the agent takes a long-short position in the two nominal bonds. Due to the high correlation between the two nominal bond prices, this can lead to extreme proportional allocations; often more than 100% of the wealth is invested in the available assets. The extreme positions are in line with the findings of Balter et al. (2021), who show that if interest rate risk and inflation risk follow a bivariate mean-reverting process, optimal allocations to bonds turn out to be extreme. Balter et al. (2021) find these results in the financial market model of Brennan and Xia (2002). Since the model of Koijen et al. (2010) is conceptually close to the model Brennan and Xia (2002), it is unsurprising that the allocations follow a similar pattern. The long-short position can be explained as follows: the agent cannot directly hedge against inflation risk and thus the only instruments available to hedge against interest rate risk and inflation risk are the two nominal bonds. Bond prices are inversely related to the interest rate and thus the nominal bond price will increase if the nominal interest rate decreases. As a consequence, the agent can hedge itself against interest rate risk by taking a long position in one nominal bond. However, the long position in the nominal bond exposes the agent to inflation risk as the price of the bond will decrease when the inflation rate increases (Balter et al., 2021). Since the agent is interested in optimizing real wealth, it will take a short position in the other available bond.

The long-short composition of nominal bond demand also gives an explanation for the extreme proportional allocations (Balter et al., 2021). While taking a longer position in one bond gives a better hedge against interest rate risk, it also increases the exposure to inflation risk, leading to extreme short positions in the second bond. Furthermore, Figure 2.1 shows that less risk averse agents will take more extreme proportional positions in the bonds, which is in line with our expectations; less risk averse agents are willing to take the risks these extreme positions bring. Linking this to the optimal strategy in Proposition 2.1, we see that the first hedge demand is constant across agents. Hence, the extreme variability in proportional allocations for less risk averse agents can be explained from the mean-variance optimal demand and the second hedge demand, which are both agent-specific. Lastly, it should be noted that a higher value of the risk aversion parameter will lead to earlier de-risking of the agent.

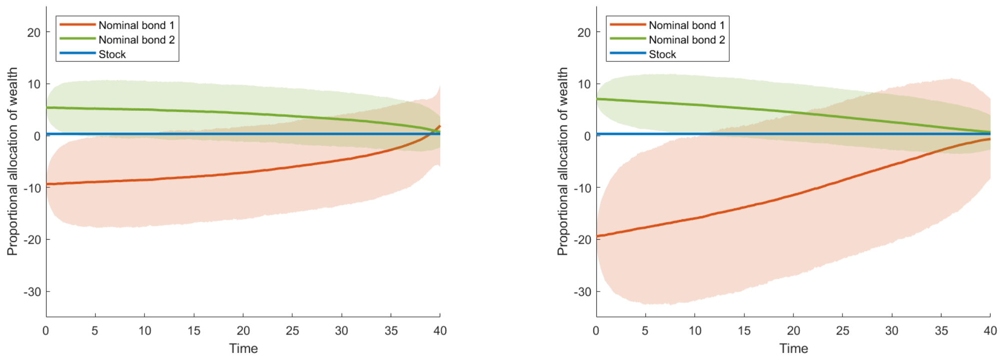
An important finding of Balter et al. (2021) is the sensitivity of the allocations with respect to the mean reversion parameters of the state variables. When the mean reversion parameters are close to each other, more extreme allocations are found. In the work of Balter et al. (2021) this is explained by the fact that interest rate risk and inflation risk are more correlated when the elements of the mean-reversion matrix are close to each other. This influences the long-short composition of the portfolio, leading to more extreme positions. We extend these results by arguing that also the price of risk can significantly influence the optimal allocations. We argue that as the norm of Λ_t increases, the possibility arises that wealth blows up to infinity, resulting in extreme allocations. To investigate the sensitivity of the allocations with respect to the parameter estimates, we present an alternative set of parameters for which numerical allocations are presented. The estimates of the mean reversion parameters and price of risk coefficients in this set of alternative parameters are half of their value in Table 2.1. Again we assume that $\tau_1 = 1$, $\tau_2 = 5$, and $\tau = 5$. It should be noted that the model contains several other parameters for which a sensitivity analysis could be performed. However, individual parameter sensitivity analysis will be too lengthy in view of the complexity of the model involving many different parameters. The choice to only investigate the sensitivity with respect to the mean reversion parameters and the price of risk is motivated by the work of Balter et al. (2021) and the findings in this thesis that the norm of the price of risk can significantly influence the allocations.

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
S_t		Λ_t		Π_t, X_t		r_t, π_t	
η_S	4.52 %	$\Lambda_{0,1}$	0.1400	$\sigma_{\Pi,1}$	0.020%	$\delta_{0,r}$	2.40%
$\sigma_{S,1}$	-0.53%	$\Lambda_{0,2}$	0.0135	$\sigma_{\Pi,2}$	$-0.568 \cdot 10^{-4}$	$\delta_{1,r(1)}$	-1.48%
$\sigma_{S,2}$	-0.76%	$\Lambda_{1,(1,1)}$	0.0745	$\sigma_{\Pi,3}$	0.610%	$\delta_{1,r(2)}$	0.53%
$\sigma_{S,3}$	-2.11%	$\Lambda_{1,(1,2)}$	-0.1905	$K_{X(1,1)}$	3.815%	$\delta_{0,\pi}$	2.00%
$\sigma_{S,4}$	16.59%	$\Lambda_{1,(2,1)}$	0.0445	$K_{X(2,1)}$	-9.500%	$\delta_{1,\pi(1)}$	-0.63%
		$\Lambda_{1,(2,2)}$	-0.0415	$K_{X(2,2)}$	17.625%	$\delta_{1,\pi(2)}$	0.14%

Table 2.2: Alternative values for the KNW parameters. Similar to Koijen et al. (2010) we define $\Lambda_{0,3} = \Lambda_{1,(3,1)} = \Lambda_{1,(3,2)} = \sigma_{\Pi,4} = 0$. In combination with the estimates for the first two elements of Λ_t and η_S this pins down the value of $\Lambda_{0,4}$, $\Lambda_{1,(4,1)}$, and $\Lambda_{1,(4,2)}$.

In Figure 2.2, we compare the strategy following from the baseline and alternative parameters for $\gamma = 5$. Again the average allocation to the two nominal bonds and stock over the investment horizon is shown as a thick line, together with the 5th and 95th percentile allocation in the shaded areas. Compared to the baseline parameters, we find similar allocations to the stock and second nominal bond. However, the alternative parameters significantly influence the allocation to the first nominal bond. First of all, we see that the average proportional allocation in the first 30 years of the horizon has become extremer. Consequently, also the percentile range around the allocation has become larger. We argue that this more extreme bond allocation can be explained by the fact that the mean reversion parameters of the state variables have changed in this example. This makes the effect of the long-short portfolio extremer, in this case resulting in an extremer short position in the first nominal bond. For $\gamma = 5$, we thus see that the effect of the mean reversion parameters is more extreme than the effect of a smaller norm of Λ_t. In

Figure A.1 we present the allocations for the alternative strategy for $\gamma = 2$, $\gamma = 8$, and $\gamma = 10$. For $\gamma = 2$ we see a similar pattern as for $\gamma = 5$, where both the average allocation and variability in the allocation to the first bond have increased significantly, whereas the allocation to the stock and second bond remains similar. For $\gamma = 8$ and $\gamma = 10$ we see that the magnitude of the allocations have become larger; also for the stock and second bond. On the other hand, the variability stays approximately the same. For $\gamma = 8$ and $\gamma = 10$ we thus see a horizontal shift of the percentile ranges. In total we conclude that the strategies are highly sensitive with respect to the mean reversion parameters and price of risk. How the parameters influence the strategy is dependent on the level of risk aversion.

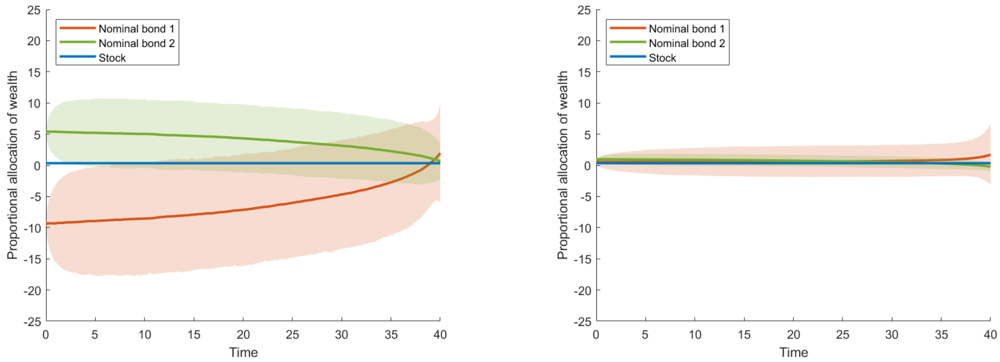


(a) Baseline parameters, $\gamma = 5$

(b) Alternative parameters, $\gamma = 5$

Figure 2.2: Comparison of the optimal strategy in the KNW model for the baseline and alternative parameters for $\gamma = 5$. The average allocations to the stock and two nominal bonds over $10,000$ simulations of the optimal investment strategy are shown in thick lines. The 90% percentile range is shown in the shaded areas. We assume $T = 40$ with monthly time steps in the simulations. Furthermore, $\tau_1 = 1$, $\tau_2 = 5$, $\tau = 5$, and $W_0 = 1$. The baseline parameters are given in Table 3.1, the alternative parameters in Table 2.2.

Another essential insight by Balter et al. (2021) is the sensitivity of the allocations with respect to the bond durations. When the bond durations are close to each other, more extreme allocations are found. Balter et al. (2021) explain this by the fact that the variance covariance matrix of the nominal bond prices becomes nearly singular when bond maturities are close to each other. Hence, the inverse of the variance matrix will converge to infinity in this case. Since the optimal strategy is dependent on the inverse of this matrix, optimal strategies can possible blow up. We argue that also the optimal strategy in the KNW model is sensitive to the bond durations. To show this, we compare the strategy for the baseline parameters for two different sets of bond durations. In Figure 2.3a, we show the strategy for the baseline parameters with $\tau_1 = 1$, $\tau_2 = 5$, and $\tau = 5$, as shown before in Figure 2.1b. In Figure 2.3b, we show the strategy for the same parameters but with $\tau_1 = 5$, $\tau_2 = 20$, and $\tau = 10$.



(a) Baseline parameters, $\gamma = 5$, $\tau_1 = 1$, $\tau_2 = 5$, and $\tau = 5$

(b) Alternative parameters, $\gamma = 5$, $\tau_1 = 5$, $\tau_2 = 20$, and $\tau = 10$

Figure 2.3: Comparison of the optimal strategy in the KNW model for the baseline parameters with $\gamma = 5$ for different bond maturities. The average allocations to the stock and two nominal bonds over $10,000$ simulations of the optimal investment strategy are shown in thick lines. The 90% percentile range is shown in the shaded areas. We assume $W_0 = 1$ and $T = 40$ with monthly time steps in the simulations. Furthermore, $\tau_1 = 1$, $\tau_2 = 5$ and $\tau = 5$ in panel a, whereas $\tau_1 = 5$, $\tau_2 = 20$ and $\tau = 10$ in panel b. The baseline parameters are given in Figure 2.1.

As can be seen above, the strategy for $\gamma = 5$ gets significantly less aggressive when the bond durations are further apart, in line with the findings of Balter et al. (2021). In Figure A.2, we present the strategy for the baseline parameters with alternative bond durations for $\gamma = 2$, $\gamma = 8$, and $\gamma = 10$. For all values of γ we see that the allocations get less extreme. However, for $\gamma = 2$ we see a relatively smaller change than for the other values of γ. This can be explained by the fact that the strategy is more extreme for less risk averse agents by construction. Another important finding is that the composition of the long-short portfolio changes for $\gamma = 2$. Whereas Figure 2.1a shows that it is optimal to take a long position in the second bond and a short position in the first bond, Figure A.2a shows exactly the opposite composition. For all other values of γ, we see that the composition of the long-short portfolio remains similar, only the magnitudes decrease.

We thus conclude that the optimal investment strategy in the model of Koijen et al. (2010) is highly sensitive with respect to the mean reversion parameters of the state variables, the price of risk, and the bond durations. Hence, we find similar results as Balter et al. (2021) found in the model of Brennan and Xia (2002). Therefore, if the KNW model is used in the pension-industry, parameter uncertainty will play an important role. Proportional allocations up to 10 times the wealth are not unique in the strategies we have studied in this paragraph. Since pension funds face a borrowing constraint in real-life, which we did not take into account in this analysis, it is unlikely that the optimal strategy can be implemented in practice without further constraints. Furthermore, the parameter sensitivity, for example with respect to bond durations, leads to other crucial problems. In real-life, nominal bonds with more than two durations are available. Since different bond durations lead to significantly different strategies, it is unclear to a pension fund in which durations they should invest. Balter et al. (2021) find that the optimal bond durations in the model of Brennan and Xia (2002) involves one medium term bond and one long term bond. However, the resulting strategy is highly sensitive with respect to the other parameters (Balter et al., 2021). Although we have not studied the optimal bond durations in the model of Koijen et al. (2010), we expect to find similar results. Hence, although the KNW model is more realistic than the Black-Scholes financial market by taking interest rate risk and inflation risk into account, it remains to be discussed how the resulting optimal strategy can be implemented by the pension-industry.

2.4 Welfare analysis

In this section, we will compare the welfare generated by the optimal investment strategy in the KNW model with the welfare generated by the mean-variance optimal strategy found by Merton (1969). Furthermore, we will investigate the welfare effects of using an incorrect risk preference parameter. Before presenting welfare losses, we must first formally define which part of the optimal strategy in Proposition 2.1 resembles the optimal strategy in the Black-Scholes financial market. We propose two possible strategies:

Definition 2.1 *Consider the optimal investment strategy in the KNW model specified in Proposition 2.1. We define the corresponding myopic Merton investment strategy as:*

$$\theta_t = \frac{1}{\gamma}\left(\Sigma^{-1}\right)^{\top}\Lambda_t$$

where Λ_t *and* Σ *are defined in* (2.6) *and* (2.16)*, respectively. The static Merton investment strategy is defined as:*

$$\theta_t = \frac{1}{\gamma}\left(\Sigma^{-1}\right)^{\top}\Lambda_0$$

where Σ *is defined in* (2.16)*.* Λ_0 *is the constant coefficient of the price of risk dynamics in* (2.6)*.*

We have seen that the optimal strategy presented in Proposition 2.1 is the composition of three portfolios: a mean-variance optimal portfolio and two hedge portfolios. The myopic strategy can be interpreted as the mean-variance optimal strategy, without taking the hedge demands into account. Note that the myopic demand is defined similar by for example Brennan and Xia (2002). The myopic strategy is determined by the time varying price of risk, which is driven by the state variables. Hence, the myopic demand will partly account for the stochasticity in state variables. In the work of Merton (1969), the price of risk is assumed to be constant and thus AZL assumes a constant price of risk when determining life cycles. Therefore it could be argued that the myopic demand takes stochasticity of the state variables into account, whereas the optimal strategy found by Merton (1969) does not. Hence, we also choose to introduce the static Merton strategy, which neglects the time variation in risk premia. We propose to take the constant coefficient from the affine structure of the price of risk as the estimate for our constant market price of risk.

An important remark is that the suboptimal strategies defined in Definition 2.1 depend on the prices of risk in the primal market, in contrast to the optimal strategy defined in Proposition 2.1, which depends on the price of risk that links the strategy in the dual market to the strategy in the primal market. This choice is explained as follows: the welfare comparison is motivated from the perspective of AZL that uses Merton's mean-variance optimal strategy rather than the optimal strategy in the model of Commissie Parameters (2022). As such, AZL generates life cycles on the basis of the prices of risk in the primal market, without any information on the optimal perturbation term that takes care of possible trading constraints. Hence, the portfolio optimization problem in the model assumed to describe the economy is not taken into account when creating life cycles; simply Merton's formula is assumed. Therefore we argue that it is more realistic to assume that any suboptimal strategy depends on the price of risk in the primal market, rather than on the price of risk that is optimal in the dual market. Note that we have assumed that, similar to Koijen et al. (2010), $\Lambda_{0,3} = \Lambda_{1,(3,1)} = \Lambda_{1,(3,2)} = 0$ and thus the trading constraint for the inflation-linked bond is automatically satisfied for both strategies in Definition 2.1.

In Table 2.3, we present the percentage welfare loss when using the suboptimal (non) myopic Merton strategy instead of the optimal strategy for both the baseline and alternative parameters. The underlying certainty equivalents and their standard errors can be found in Table A.1.

γ	2	3	4	5	6	7	8	9	10
Myopic baseline (%)	-21.54	-28.12	-30.21	-31.65	-33.13	-34.72	-36.46	-38.21	-39.96
	(2.00)	(2.79)	(2.97)	(3.02)	(2.99)	(2.89)	(2.76)	(2.63)	(2.52)
Myopic alternative (%)	-10.96	-17.94	-24.02	-30.11	-36.02	-41.45	-46.20	-50.22	-53.57
	(0.97)	(1.97)	(2.57)	(2.89)	(3.07)	(3.12)	(3.01)	(2.78)	(2.52)
Static baseline (%)	-90.78	-82.19	-75.74	-71.18	-68.04	-65.96	-64.65	-63.90	-63.57
	(0.56)	(1.25)	(1.59)	(1.70)	(1.70)	(1.65)	(1.57)	(1.49)	(1.43)
Static alternative (%)	-67.45	-57.09	-53.67	-54.02	-56.35	-59.45	-62.49	-65.13	-67.29
	(0.92)	(1.60)	(1.89)	(2.08)	(2.38)	(2.62)	(2.66)	(2.55)	(2.37)

Table 2.3: Rounded percentage loss in certainty equivalent when using the suboptimal strategies defined in Definition 2.1 instead of the optimal strategy defined in Proposition 2.1, for the baseline and alternative parameters. The baseline parameters can be found in Table 2.1, the alternative parameters in Table 2.2. Percentage standard errors are reported in parentheses. The data is calculated over 10,000 simulations. We assume $T = 40$ with monthly time steps in the simulations, $W_0 = 1$, $\tau_1 = 1$, $\tau_2 = 5$, and $\tau = 5$.

For both the baseline and alternative parameters we find the welfare losses to be smaller for the myopic Merton strategy than for the static Merton strategy. This is in line with our expectations since the myopic strategy takes the time variation in prices of risk into account and thus, implicitly, it takes the evolution of the state variables into account. As the state variables have a central role in the model setup, accounting for their evolution will benefit the agent. The static strategy does not take the time variation in prices of risk into account, while the evolution of the assets is still influenced by this time variation, leading to severe welfare losses for both sets of parameter estimates. Another way to interpret the relatively small welfare losses of the myopic strategy (as compared to the static strategy) lies in an analytical explanation of the optimal strategy in Proposition 2.1. The magnitude of the first hedge demand is determined by σ_Π, whose estimated values are small compared to the values of Λ_t and $\tilde{H}(t, X_t)$. Hence, neglecting the first hedge demand will not significantly influence the portfolio composition. Furthermore, the magnitude of the second hedge demand is determined by $\tilde{H}(t, X_t)$, which only influences the position in the two nominal bonds. Hence, the only difference between the optimal strategy and the myopic strategy is the demand for the two nominal bonds to hedge against fluctuations in the state variables. The static demand differs from the optimal strategy in an extra way; namely by not taking the time variation in the price of risk into account. This thus leads to higher welfare losses.

For the static Merton strategy we find, for the baseline parameters, that an agent with lower risk aversion faces smaller welfare losses. It should be noted that, although we find this pattern, we cannot conclude a statistically significant difference between the welfare losses for $\gamma = 7$ and higher. The approximately inverse relation of the welfare losses with the price of risk can be explained by the fact that an agent with lower risk aversion takes more extreme positions. Hence, a larger variation in the price of risk is needed to mimic the optimal position. The static Merton strategy takes no time variation in the price of risk into account and thus the effect will be extremer for smaller values of γ. For the alternative parameters we find a different pattern than for the static strategy. Until $\gamma = 6$ we see, taking standard errors into account, a decrease in the welfare loss as the agent gets more risk averse. However, from $\gamma = 7$ we see an increase in the welfare loss. This is in line with the strategies we show for $\gamma = 8$ and $\gamma = 10$ in Figure A.1. Here we see, as compared to Figure 2.1, that the allocations have become extremer for the highest values of γ when using the alternative parameters instead of the baseline parameters. We thus see that these more extreme allocations lead to higher welfare losses.

For the myopic Merton strategy we find different patterns than for the static strategy. For the baseline parameters we find that the welfare losses increase as γ increases, although we cannot conclude a statistical significant difference between the welfare losses for $\gamma = 7$ and higher. For the alternative parameters we also find that the welfare losses increase with γ. For those parameters we can conclude that the welfare losses are significantly different for different values of the risk aversion parameter. We argue that the proportional relationship between γ and the welfare loss for the myopic strategy can be explained from the composition of the optimal portfolio. Both in the static and myopic strategy, the hedge demands are not taken into account. However, the static strategy also neglects the time variation in the price of risk. Since lower values of γ lead to more variability in the optimal allocation, the time variation in the price of risk is needed to account for the variability induced by the optimal strategy. Hence, the effect of neglecting the time variation in risk premia is dominant over the effect of not taking the hedge demands into account, for the static strategy. However, for the myopic strategy the time variation in risk premia is taken into account. Hence, the only difference with the optimal portfolio is that the hedge demands are not taken into account. We conclude that purely the effect of the hedge demand affects the more risk averse agents harder. In other words, a more risk averse agent puts relatively more weight of the portfolio in hedging than less risk averse agents. Hence, the relative share of the mean-variance optimal demand in the optimal portfolio is higher for less risk averse agents. Consequently, the welfare losses for the myopic strategy increase with γ. Overall we conclude that the the myopic strategy leads to significantly smaller welfare losses than the static strategy because the evolution of the price of risk is taken into account. However, the welfare losses for both the myopic and static strategy are highly sensitive to the parameters used.

We want to compare the magnitude of the welfare losses from using a suboptimal strategy to the magnitude of the welfare losses from using an incorrect risk preference parameter. We present the welfare losses of using an incorrect risk preference parameter for the baseline and alternative parameters in Table 2.4 and Table 2.5, respectively. The corresponding certainty equivalents and their standard errors can be found in Table A.2 and Table A.3, respectively. The tables with welfare losses can be interpreted as follows: on the vertical axis the agent's true risk preference parameter, γ_{true}, is reported whereas the horizontal axis shows possible measured values of the risk preference parameter, γ_{measured}. For each cell in Table 2.4 and Table 2.5 the strategy is calculated on the basis of γ_{measured}, whereas the strategy is evaluated on the basis of γ_{true}[15]. When $\gamma_{\text{measured}} \neq \gamma_{\text{true}}$ the strategy is suboptimal, which in general leads to welfare losses. Further information on how the certainty equivalents and their standard errors are calculated can be found in Section 1.3.

[15]Table A.2 and Table A.3 show the certainty equivalents for the suboptimal strategy where an agent invests according to γ_{measured}, but evaluates the strategy according to γ_{true}. The certainty equivalents of the optimal strategies based on γ_{true} can be found in Table A.1. Combining Table A.2 and Table A.3 with Table A.1 thus leads to the welfare losses presented in Table 2.4 and Table 2.5.

True γ \ Measured γ	2	3	4	5	6	7	8	9	10
2	–	-19.85	-48.84	-64.97	-73.99	-79.44	-82.98	-85.41	-87.16
		(3.31)	(2.60)	(1.90)	(1.46)	(1.17)	(0.98)	(0.85)	(0.75)
3	-49.03	–	-7.43	-23.90	-37.35	-47.19	-54.35	-59.67	-63.72
	(6.51)		(3.86)	(4.20)	(3.79)	(3.33)	(2.95)	(2.64)	(2.39)
4	-71.84	-17.91	–	-4.11	-13.63	-22.94	-30.81	-37.21	-42.40
	(4.44)	(5.56)		(3.06)	(3.96)	(4.04)	(3.87)	(3.63)	(3.40)
5	-79.91	-34.13	-7.04	–	-2.75	-8.74	-15.17	-21.11	-26.30
	(2.89)	(6.31)	(3.78)		(2.29)	(3.30)	(3.65)	(3.70)	(3.63)
6	-83.55	-43.96	-15.80	-2.95	–	-2.03	-6.07	-10.65	-15.12
	(2.10)	(5.55)	(5.21)	(2.60)		(1.72)	(2.66)	(3.12)	(3.31)
7	-85.53	-49.92	-22.70	-7.66	-1.22	–	-1.58	-4.47	-7.83
	(1.64)	(4.76)	(5.34)	(3.93)	(1.86)		(1.32)	(2.15)	(2.63)
8	-86.75	-53.78	-27.71	-12.05	-3.84	-0.44	–	-1.27	-3.43
	(1.35)	(4.13)	(5.07)	(4.40)	(2.97)	(1.38)		(1.03)	(1.74)
9	-87.57	-56.43	-31.35	-15.65	-6.61	-1.92	-0.06	–	-1.05
	(1.14)	(3.63)	(4.72)	(4.46)	(3.50)	(2.28)	(1.05)		(0.83)
10	-88.15	-58.36	-34.06	-18.51	-9.10	-3.69	-0.91	0.13	–
	(0.99)	(3.24)	(4.38)	(4.36)	(3.72)	(2.79)	(1.78)	(0.83)	

Table 2.4: Rounded percentage loss in certainty equivalent when the agent invests according to an incorrect risk aversion parameter instead of the true risk preference parameter. Percentage standard errors are reported in parentheses. The data is calculated over $10,000$ simulations. We assume $T = 40$ with monthly time steps in the simulations, $W_0 = 1$, $\tau_1 = 1$, $\tau_2 = 5$, and $\tau = 5$. Other parameter values are given in Table 2.1.

True γ \ Measured γ	2	3	4	5	6	7	8	9	10
2	–	-13.53	-32.58	-45.12	-53.28	-58.84	-62.80	-65.75	-68.02
		(1.66)	(1.60)	(1.40)	(1.24)	(1.12)	(1.03)	(0.96)	(0.90)
3	-39.88	–	-3.59	-13.46	-22.23	-29.16	-34.56	-38.82	-42.23
	(7.77)		(2.31)	(2.63)	(2.56)	(2.42)	(2.28)	(2.16)	(2.05)
4	-71.40	-18.09	–	-0.75	-6.00	-11.65	-16.70	-21.00	-24.62
	(5.75)	(5.74)		(2.19)	(2.79)	(2.91)	(2.88)	(2.81)	(2.72)
5	-82.63	-39.64	-9.78	–	0.21	-2.72	-6.38	-9.97	-13.25
	(3.13)	(7.11)	(3.95)		(1.86)	(2.58)	(2.84)	(2.91)	(2.90)
6	-87.28	-53.42	-22.96	-6.03	–	0.54	-1.13	-3.55	-6.12
	(1.95)	(5.67)	(5.89)	(2.82)		(1.55)	(2.28)	(2.61)	(2.75)
7	-89.68	-61.49	-33.80	-14.47	-4.08	–	0.65	-0.31	-1.93
	(1.35)	(4.32)	(5.74)	(4.57)	(2.09)		(1.28)	(1.98)	(2.35)
8	-91.10	-66.52	-41.56	-22.37	-9.82	-2.95	–	0.67	0.14
	(1.00)	(3.39)	(4.99)	(4.99)	(3.55)	(1.61)		(1.07)	(1.71)
9	-92.03	-69.88	-47.05	-28.76	-15.59	-7.06	-2.24	–	0.64
	(0.79)	(2.75)	(4.26)	(4.75)	(4.16)	(2.81)	(1.27)		(0.90)
10	-92.68	-72.27	-51.05	-33.72	-20.64	-11.38	-5.32	-1.77	–
	(0.64)	(2.29)	(3.66)	(4.32)	(4.21)	(3.44)	(2.26)	(1.03)	

Table 2.5: Rounded percentage loss in certainty equivalent when the agent invests according to an incorrect risk aversion parameter instead of the true risk preference parameter. Percentage standard errors are reported in parentheses. The data is calculated over $10,000$ simulations. We assume $T = 40$ with monthly time steps in the simulations, $W_0 = 1$, $\tau_1 = 1$, $\tau_2 = 5$, and $\tau = 5$. Other parameter values are given in Table 2.2.

First of all, it should be noted that some percentages presented in Table 2.4 and Table 2.5 are positive. Although using a suboptimal strategy can never result in a welfare gain, positive numbers can be found as a consequence of the numerical implementation. When the measured risk aversion parameter is close to its true value, the numerical implementations will lead to approximately similar certainty equivalents. As a consequence of the simulations, the certainty equivalent of the suboptimal strategy can then be slightly higher than the optimal CE. However, in those cases no statistical significant difference between the two certainty equivalents can be concluded. In other words, the welfare loss is not significantly different from zero and thus the strategies lead to approximately the same wealth.

For both the baseline and alternative parameters we find, in line with our expectations, that the welfare losses increase as the measured value of γ is further away from the true value. Furthermore, we see that an overestimation of γ of at most one leads to small welfare losses, which are in many cases not statistically significant different from zero. This is only not the case for $\gamma_{\text{true}} = 2$. In these case, even using the strategy of $\gamma_{\text{measured}} = 3$ will lead to a significant welfare loss. This can be explained by the fact that strategy for $\gamma = 2$ is so aggressive that even a little deviation will lead to welfare losses. Furthermore, we see that for the alternative parameters also an overestimation of size two in general do not lead to large welfare losses. This is not the case for the baseline parameters where an overestimation of size two can cause significant welfare losses. This again shows the sensitivity of the model with respect to the parameter values.

In general the welfare effects of a relatively small misestimate are smaller for higher values of γ_{true}. This can again be explained by the fact that the strategies become less aggressive for higher values of γ. Furthermore, the effects of a small underestimation of the risk preference parameter are generally larger than the effect of a small overestimation. A possible explanation for this is that lower values of the risk aversion parameter lead to more extreme allocations and thus an inaccurate value of γ will lead to larger deviations from the optimal strategy. In general, we find significant differences between the magnitudes of the losses for the baseline and alternative parameters, in line with the losses arising from the use of the myopic or static strategy. Welfare losses are larger for the baseline parameters, which could possibly be explained by the larger norm of the price of risk, which results in more extreme allocations. Hence, it is likely that more extreme welfare losses are found. Although the magnitudes of the welfare losses differ per set of parameter estimates, approximately similar patterns are found. Welfare losses increase as the measured risk aversion parameter deviates more from the true risk aversion parameter, for both the baseline and alternative parameters. Acting as if the agent is less risk averse than it actually is leads to more extreme welfare losses than overestimation of the risk preference parameter.

We have thus found that assuming a constant price of risk leads to extreme welfare losses in the KNW model. When the affine structure of the price of risk is taken into account these welfare effects can to a large extend be mitigated. With respect to using an incorrect risk aversion parameter, we find that absolute deviations from the true risk aversion parameter of at most two will lead to relatively small welfare losses, at most equal to the losses when using the myopic Merton strategy. For larger differences between the measured and true γ, welfare losses will increase significantly. In general, the effects are larger when the risk aversion of an agent is underestimated. This is explained by the fact that low values of the risk aversion parameter lead to extreme allocations, and thus welfare losses will be larger when the risk aversion parameter is underestimated.

Chapter 3

Stochastic volatility

In the previous chapter we examined the effect of stochastic interest rates and inflation on the optimal portfolio, whereas we assumed the volatility of the stock to be constant. However, empirical evidence shows that the volatility of asset prices fluctuates over time. Furthermore, volatility risk significantly influences asset prices (Huang et al., 2019). Therefore we will investigate the welfare effects of stochastic volatility in this chapter. We take the model introduced by Heston (1993) to describe the stochastic volatility framework. In Section 3.1 we will present the financial market, combined with the dynamics of an asset in which the agent cannot invest. This asset fulfills the same function as the inflation-linked bond in Chapter 2; in the dual market we will lift the trading constraint on the asset so that we can apply the Martingale method to our portfolio optimization problem. In Section 3.2 we will solve the dual optimization problem and link the corresponding investment strategy to a strategy in the original incomplete market. In Section 3.3 we will exploit the properties of the optimal strategy by a numerical analysis. Finally, Section 3.4 will provide the welfare losses of using Merton's suboptimal mean-variance strategy and the welfare losses resulting from using a wrong risk preference parameter.

In contrast to other stochastic volatility models, the model by Heston (1993) allows for correlation between asset returns and the volatility process. Heston (1993) uses the model to find closed form prices for call options on stocks with stochastic volatility and thus no portfolio optimization problem is analyzed. Consequently, no welfare analysis is performed. However, the portfolio optimization problem in Heston's model is performed by several other authors, for example by Liu and Pan (2003), Nielsen and Jönsson (2015), Chen et al. (2018), and Yang and Pelsser (2023). Liu and Pan (2003), Chen et al. (2018), and Yang and Pelsser (2023) deal with the incompleteness of the model by adding an extra asset to the market. In this complete market an optimal strategy is found. However, the optimal strategy in the complete market is not linked to the optimal strategy in the incomplete market. Hence, we extend the results of Liu and Pan (2003), Chen et al. (2018), and Yang and Pelsser

(2023) by presenting a strategy that is optimal in the original incomplete market. Nielsen and Jönsson (2015) solve the portfolio problem using a dynamic programming approach and thus a strategy optimal in the incomplete market is found. The use of a dynamic programming method introduces the need for complex verification arguments in the work of Nielsen and Jönsson (2015). The same holds for Liu and Pan (2003), who also solve the optimization problem with a dynamic method. Chen et al. (2018) and Yang and Pelsser (2023) rely on Laplace transforms when solving the optimization problem. We solve the portfolio optimization problem using Martingale method in combination with duality theory. We argue that this approach is technically less demanding than the approaches used by Liu and Pan (2003), Nielsen and Jönsson (2015), Chen et al. (2018), and Yang and Pelsser (2023). Hence, next to finding the optimal strategy in the incomplete market, we also depart from the work in the literature by using a different and arguably simpler method.

Liu and Pan (2003) focus on an analysis of the optimal strategy, particularly when the volatility process is allowed to jump[16]. Furthermore, they investigate the effect of not taking stochastic volatility into account. It is found that no significant welfare loss arises when volatility risk is not hedged. Nielsen and Jönsson (2015) perform a similar welfare analysis and also find that no significant welfare gains are found when volatility risk is hedged. This is in line with the results we will present in Section 3.4. Although Chen et al. (2018) and Yang and Pelsser (2023) solve the optimal investment problem in Heston's model, none of these papers use the optimal strategy to perform a welfare analysis. Chen et al. (2018) compare the optimal strategy in Heston's model with strategies where extra constraints are imposed on the portfolio. Yang and Pelsser (2023) examine the effects of using different utility functions than the CRRA utility in a stochastic volatility framework.

We end the introduction of this chapter with a brief remark on the work of Kraft (2005), who proposes verification results for the optimal strategies when using a dynamic programming approach. Dynamic programming approaches do not take care of the verification of constraints in itself. Therefore, an ex-post verification of the optimal strategy is important. Kraft (2005) argues that several papers that study optimal investment in a stochastic volatility framework have not taken this verification into account. For example, the optimal strategy found by Liu (2001) is not unique. Hence, the work of Kraft (2005) plays an important role when considering stochastic volatility in combination with a dynamic programming approach. Since we use the Martingale method the verification of the strategies is taken care of within the method itself.

[16]Jumps in the volatility process make it possible to capture sudden market movements more realistically. See Branger et al. (2008) for more information on optimal investment in a stochastic volatility model in which the volatility is allowed to jump.

3.1 Financial market

In this section, we present the financial market introduced by Heston (1993). The model is an extension of the Black-Scholes financial market where the volatility of the stock is allowed to be stochastic. The agent can invest in a stock and a money market account, whose dynamics are given as follows:

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu_t dt + \sqrt{\nu_t} dZ_t^1, \\ \frac{dB_t}{B_t} &= r dt\end{aligned} \tag{3.1}$$

where Z_t^1 is a one dimensional standard Brownian motion, $S_0 = B_0 = 1$, and $\mu_t = r + \eta_1 \nu_t$ with ν_t the stochastic volatility as defined in (3.2). We assume $\eta_1 \in \mathbb{R}$ so that η_1 pins down the price of risk of Z_t^1, i.e. $\frac{\mu_t - r}{\sqrt{\nu_t}} = \eta_1 \sqrt{\nu_t}$[17]. As opposed to Chapter 2, the interest rate is constant and we do not allow for investments in bonds, similar to Heston (1993). The stochastic variance is modeled according to a mean reverting process:

$$d\nu_t = \kappa(\bar{\nu} - \nu_t)dt + \delta\sqrt{\nu_t}(\rho dZ_t^1 + \sqrt{1-\rho^2} dZ_t^2) \tag{3.2}$$

where Z_t^2 is a one-dimensional standard Brownian motion, independent of Z_t^1. Similar to Z_t^1, we assume that the price of risk of the second Brownian motion is proportional to the square root of the stochastic variance process, i.e. the price of risk of Z_t^2 equals $\eta_2\sqrt{\nu_t}$, where $\eta_2 \in \mathbb{R}$. We can assume a price of risk of the second Brownian motion in the original market because duality theory forces an asset to be defined that is driven by Z_t^2. Without this added asset, no price of risk of the second Brownian motion would exist. A dependence structure between Z_t^1 and Z_t^2 is created using a Cholesky decomposition with correlation coefficient ρ. The starting value of the stochastic variance process ν_0 is one of the parameters we set when simulating the model. We see that the variance fluctuates around its long run mean $\bar{\nu}$ with mean reversion speed κ. δ can be interpreted as the volatility of the variance process. To guarantee positivity of the variance we impose the restrictions $2\kappa\bar{\nu} \geq \delta^2$ and $\nu_0 > 0$ (Chen et al., 2018).

Combining the two prices of risk leads to the following price of risk vector for the original market:

$$\Lambda_t = \begin{bmatrix} \eta_1\sqrt{\nu_t} \\ \eta_2\sqrt{\nu_t} \end{bmatrix} \tag{3.3}$$

[17] The assumption that the prices of risk are proportional to the square root of the stochastic variance is in line with the work of Heston (1993). Without this assumption no closed form solution to the portfolio optimization problem could exist. In this thesis, the conditional expectation in (B.7) would no longer be affine in ν_t without this assumption. Consequently, no closed form solution to the corresponding conditional expectation would exist in this case.

Since we only allow for investments in a stock and a money market account, the Heston's financial market does not provide the possibility to take financial positions in products that are directly driven by the second Brownian motion Z_t^2. As such the financial market is incomplete. We deal with this market incompleteness using the primal-dual approach explained in Section 1.2. Hence, we introduce an extra asset to the original market in which the agent cannot invest and thus we maintain market incompleteness in the original market. In the dual market the trading constraint is lifted and thus Z_t^2 can be traded. Since the added asset is driven by the untraded Brownian motion, the price of risk in the dual market of the Brownian motion untraded in the original market equals η_2 plus the perturbation term, which allows us to link the strategy in the dual market to the strategy in the primal market.

Before defining the extra asset, we first propose the pricing kernel. By virtue of using the money market account B_t as numéraire, the pricing kernel in the market with an added asset is given as follows:

$$\begin{aligned} \frac{d\phi_t}{\phi_t} &= -rdt - (\eta_1\sqrt{\nu_t}dZ_t^1 + \eta_2\sqrt{\nu_t}dZ_t^2) \\ &= -rdt - \Lambda_t' dZ_t \end{aligned} \tag{3.4}$$

with $\phi_0 = 1$.

By means of this pricing kernel we can define the extra asset we introduce in the original market. In line with Chen et al. (2018), we let the price of this option be equal to $O_t = h(t, S_t, \nu_t)$. We then know that its price is defined as follows:

$$O_t = \mathbb{E}\left[\frac{\phi_T}{\phi_t} h(t, S_t, \nu_t) \Big| \mathcal{F}_t\right] \tag{3.5}$$

Using that $\phi_t O_t$ is a martingale, we find that O_t adheres to the following dynamics (see Appendix B.1 for detailed derivations):

$$\begin{aligned} dO_t = &\Big(rO_t + (h_S S_t + h_\nu \delta\rho)\eta_1\nu_t + h_\nu\delta\sqrt{1-\rho^2}\eta_2\nu_t\Big)dt \\ &+ \sqrt{\nu_t}\Big((h_S S_t + h_\nu\delta\rho)dZ_t^1 + h_\nu\delta\sqrt{1-\rho^2}dZ_t^2\Big) \end{aligned} \tag{3.6}$$

We stress that the agent is not allowed to trade in O_t in the original market. Hence, although we use an exact postulate for O_t, any asset influenced by the stochastic volatility would satisfy the requirements on the added asset imposed by duality theory.

To support further derivations, we combine the dynamics of all assets in the original market in the vector $Y_t \in \mathbb{R}^{2\times 1}$, which then reads as follows:

$$Y_t = \begin{bmatrix} S_t \\ O_t \end{bmatrix} \tag{3.7}$$

The dynamics of Y_t are specified as:

$$dY_t = \text{diag}(Y_t)\Big((r + \Sigma_t \Lambda_t)dt + \Sigma_t dZ_t\Big) \tag{3.8}$$

where $Z_t = \begin{bmatrix} Z_t^1 & Z_t^2 \end{bmatrix}'$. Furthermore, $\Sigma_t \in \mathbb{R}^{2\times 2}$ represents the time dependent variance-covariance matrix of the financial market:

$$\Sigma_t = \begin{bmatrix} \sqrt{\nu_t} & 0 \\ \frac{\sqrt{\nu_t}(h_s S_t + \delta\rho h_\nu)}{O_t} & \frac{\sqrt{\nu_t} h_\nu \delta \sqrt{1-\rho^2}}{O_t} \end{bmatrix} \tag{3.9}$$

3.2 Portfolio optimization problem

In this section we solve the portfolio optimization problem in Heston's stochastic volatility model. In the original incomplete market, the agent faces the following optimization problem:

$$\begin{aligned} &\sup_{\theta} \ \mathbb{E}\Big[\frac{W_T^{1-\gamma}}{1-\gamma}\Big] \\ &\text{s.t. } dW_t = W_t(r + \theta_t' \Sigma_t \Lambda_t)dt + W_t \theta_t \Sigma_t dZ_t \end{aligned} \tag{3.10}$$

where $\theta_t \in \mathbb{R}^{2\times 1}$ represents an investment strategy in the original market. Due to the trading constraint on O_t, $\theta_{2,t}$ should equal zero for each time point. Therefore, the optimization problem in (3.10) cannot be solved with the Martingale method: the corresponding static problem will be characterized by infinitely many pricing kernels. To apply the Martingale method, we find the optimal investment strategy in the original market via the dual market. In the dual market the trading constraint on O_t is lifted. The asset dynamics in the dual market are equal to the dynamics in the primal market, except for the price of risk of the Brownian motion untraded in the original market. To satisfy the mathematical equivalency between the primal and dual solution we set the price of risk of the second Brownian motion in the dual market equal to the sum of η_2 and the time-dependent but non-stochastic perturbation term a_t. Hence, in line with the duality theory, the price of risk in the dual market is defined as follows:

$$\tilde{\Lambda}_t = \begin{bmatrix} \eta_1 \sqrt{\nu_t} \\ \tilde{\eta}_{2,t}\sqrt{\nu_t} \end{bmatrix} = \begin{bmatrix} \eta_1 \sqrt{\nu_t} \\ (\eta_2 + a_t)\sqrt{\nu_t} \end{bmatrix} \tag{3.11}$$

where η_1 and η_2 are the prices of risk corresponding to the primal market defined in (3.3) and a_t is the perturbation term. We find the optimal strategy in Heston's model to be time-dependent but not stochastic. Hence, only a time-dependent perturbation term is needed to make the dual strategy admissible in the primal market. The modification of the price of risk of the untraded Brownian motion happens in such a way that the choice of a_t that reduces the demand of O_t to zero corresponds to minimizing the dual objective, which in turn maximizes the primal objective (Cvitanic and Karatzas, 1992).

As a consequence of working with a modified price of risk in the dual market the dynamics of the pricing kernel change. Although the dynamics of the pricing kernel in the dual market trivially follow from the modified price of risk, we choose to present it to ensure clarity:

$$\frac{d\tilde{\phi}_t}{\tilde{\phi}_t} = -rdt - (\eta_1\sqrt{\nu_t}dZ_t^1 + \tilde{\eta}_{2,t}\sqrt{\nu_t}dZ_t^2) \tag{3.12}$$

where $\tilde{\phi}_0 = 1$.

In the dual market we face a new optimization problem. By choosing $\tilde{\theta}_t(a)$ optimally, the agent aims at maximizing terminal dual wealth. This leads to the following dynamic optimization problem:

$$\begin{aligned} &\sup_{\theta}\ \mathbb{E}\Big[\frac{\tilde{W}_T^{1-\gamma}}{1-\gamma}\Big] \\ &\text{s.t. } d\tilde{W}_t = \tilde{W}_t(r + \tilde{\theta}_t(a)'\Sigma_t\tilde{\Lambda}_t)dt + \tilde{W}_t\tilde{\theta}_t(a)'\Sigma_t dZ_t \end{aligned} \tag{3.13}$$

Note that the budget constraint in (3.13) specifies the evolution of *dual* wealth. Hence, $\theta_t(a) \in \mathbb{R}^{2\times 1}$ describes an investment strategy in the dual market, which is dependent on the perturbation term a_t. The first element of $\theta_t(a)$ corresponds to the position in the dual market in the stock. The second element corresponds to the position in O_t, which we will set to zero by determining the appropriate perturbation term in the dual market. The remainder, $1 - \sum_{i=1}^{2}\theta_{i,t}(a)$, is invested in the money market account. The optimal strategy in the primal market will depend on the optimal perturbation term and thus we will denote this strategy by $\theta_t(a^*)$. The budget constraint states that the agent earns the risk premium $\Sigma_t\tilde{\Lambda}_t$ on the fraction of wealth that is invested in the risky asset. The remainder is invested in the money market account, paying the short rate r. Once we have made the choice of the perturbation term that reduces the demand of O_t to zero, we can specify the evolution of *primal* wealth from the budget constraint. Similarly as for the KNW model, it should be noted that the optimal dual wealth is equal to the optimal primal wealth after inserting the optimal a_t (Cvitanic and Karatzas, 1992).

Following the logic of Cox and Huang (1989, 1991) and Munk (2017), we transfer this dynamic problem to its static counterpart:

$$\begin{aligned} &\sup_{\tilde{W}_T}\ \mathbb{E}\Big[\frac{\tilde{W}_T^{1-\gamma}}{1-\gamma}\Big] \\ &\text{s.t. } \mathbb{E}\Big[\tilde{\phi}_T\tilde{W}_T\Big] = W_0 \end{aligned} \tag{3.14}$$

Since the agent now faces a static optimization problem, we resolve to a Lagrangian approach. We find the following Lagrangian $\mathcal{L}$ for some Lagrange multiplier l:

$$\begin{aligned}\mathcal{L} &= \mathbb{E}\Big[\frac{\tilde{W}_T^{1-\gamma}}{1-\gamma}\Big] + l\Big(W_0 - \mathbb{E}\big[\tilde{\phi}_T\tilde{W}_T\big]\Big) \\ &= \mathbb{E}\Big[\frac{\tilde{W}_T^{1-\gamma}}{1-\gamma} - l\tilde{\phi}_T\tilde{W}_T\Big] + lW_0\end{aligned} \tag{3.15}$$

Taking derivatives inside the expectation and rewriting for $\tilde{W}_T^*$ leads to the following expression for optimal terminal dual wealth:

$$\tilde{W}_T^* = (l\tilde{\phi}_T)^{-\frac{1}{\gamma}} \tag{3.16}$$

Plugging this result in the static budget constraint from (3.14) gives us an analytical expression for the Lagrange multiplier:

$$l = W_0^{-\gamma}\left(\mathbb{E}\Big[(\tilde{\phi}_T)^{1-\frac{1}{\gamma}}\Big]\right)^{\gamma} = W_0^{-\gamma}g_T^{\gamma} \tag{3.17}$$

where we have defined g_T as $\mathbb{E}\Big[(\tilde{\phi}_T)^{1-\frac{1}{\gamma}}\Big]$. Substituting the expression for the Lagrange multiplier back in (3.16) leads to the following expression for optimal terminal dual wealth:

$$\tilde{W}_T^* = \frac{W_0}{g_T}(\tilde{\phi}_T)^{-\frac{1}{\gamma}} \tag{3.18}$$

By using the Martingale method we treat wealth as a traded asset and thus we can find the optimal dual wealth at time t by discounting the optimal dual terminal wealth with the appropriate pricing kernel:

$$\begin{aligned}\tilde{W}_t^* &= \frac{1}{\tilde{\phi}_t}\mathbb{E}\left[\tilde{\phi}_T\tilde{W}_T^*|\mathcal{F}_t\right] \\ &= \frac{W_0}{g_T}(\tilde{\phi}_t)^{-\frac{1}{\gamma}}\mathbb{E}\left[\left(\frac{\tilde{\phi}_T}{\tilde{\phi}_t}\right)^{\frac{\gamma-1}{\gamma}}\middle|\mathcal{F}_t\right]\end{aligned} \tag{3.19}$$

Similar to the optimization problem in the KNW model, we need to find an explicit expression for $\mathbb{E}\left[\left(\frac{\tilde{\phi}_T}{\tilde{\phi}_t}\right)^{\frac{\gamma-1}{\gamma}}\middle|\mathcal{F}_t\right]$. In Appendix B.2 we show that $(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}$ is an exponentially affine function of the stochastic variance and thus we can solve the conditional expectation by the results of Duffie and Kan (1996) on affine yield models. Therefore we find the following expression for the conditional expectation of interest:

$$\tilde{P}(t,\nu_t) = \mathbb{E}\left[\left(\frac{\tilde{\phi}_T}{\tilde{\phi}_t}\right)^{\frac{\gamma-1}{\gamma}}\middle|\mathcal{F}_t\right] = \exp\left(\tilde{A}(t) + \tilde{B}(t)\nu_t\right) \tag{3.20}$$

where $\tilde{A}(t) \in \mathbb{R}$ and $\tilde{B}(t) \in \mathbb{R}$. We find the following system of ODE's that describes $\tilde{A}(t)$ and $\tilde{B}(t)$:

$$\begin{cases} \dot{\tilde{A}}(t) = -\kappa\bar{\nu}\tilde{B}(t) + \frac{\gamma-1}{\gamma}r \\ \dot{\tilde{B}}(t) = -\frac{1}{2}\delta^2\tilde{B}(t)^2 + \Big(\kappa + \frac{\gamma-1}{\gamma}\delta(\rho\eta_1 + \sqrt{1-\rho^2}\tilde{\eta}_{2,t})\Big)\tilde{B}(t) \\ \qquad + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\eta_1^2 + \tilde{\eta}_{2,t}^2) \end{cases} \tag{3.21}$$

By combining (3.18) and (3.19) we know that $\mathbb{E}\Big[(\frac{\tilde{\phi}_T}{\tilde{\phi}_t})^{\frac{\gamma-1}{\gamma}}\Big|\mathcal{F}_t\Big] = 1$ and thus we find the terminal conditions $\tilde{A}(T) = \tilde{B}(T) = 0$. For derivations of (3.20) and (3.21) we refer to Appendix B.2. The ODE describing $\tilde{B}(t)$ is a one-dimensional Riccati equation with constant coefficients and thus we know it has an analytical solution. Note however that the system above describes dual wealth and hence it is dependent on the perturbation term a_t. Once we have made the choice of a perturbation term that reduces the demand of O_t to zero, we can present the system of ODE's specific to the evolution of the optimal primal wealth. We therefore choose to present the solution to the system in (3.21) once we have determined a_t, so that only the solution specific to the optimal perturbation term is given.

By applying the Martingale method we treat wealth as a traded asset. Therefore we know that the optimal wealth, discounted with the pricing kernel, should be a martingale. First, note that from (3.20) we find the following for $\tilde{W}_t^*\tilde{\phi}_t$:

$$\begin{aligned} \tilde{W}_t^*\tilde{\phi}_t &= \frac{W_0}{g_T}(\tilde{\phi}_t)^{-\frac{1}{\gamma}}\mathbb{E}\left[\left(\frac{\tilde{\phi}_T}{\tilde{\phi}_t}\right)^{\frac{\gamma-1}{\gamma}}\middle|\mathcal{F}_t\right]\tilde{\phi}_t \\ &= \frac{W_0}{g_T}(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}\tilde{P}(t,\nu_t) \end{aligned} \tag{3.22}$$

whose dynamics are defined as follows (see Appendix B.2 for derivations):

$$\begin{aligned} d\tilde{W}_t^*\tilde{\phi}_t = \tilde{W}_t^*\tilde{\phi}_t\Big(&(\tilde{B}(t)\delta\rho - \frac{\gamma-1}{\gamma}\eta_1)\sqrt{\nu_t}dZ_t^1 \\ &+ (\tilde{B}(t)\delta\sqrt{1-\rho^2} - \frac{\gamma-1}{\gamma}\tilde{\eta}_{2,t})\sqrt{\nu_t}dZ_t^2\Big) \end{aligned} \tag{3.23}$$

Note that the dynamic budget constraint in (3.13) provides an alternative formulation for the evolution of the optimal dual wealth. We know that also the product of this specification of wealth and the pricing kernel should be a martingale. Hence, we find the following alternative formulation of $\tilde{W}_t^*\tilde{\phi}_t$ in Appendix B.2:

$$d\tilde{W}_t^*\tilde{\phi}_t = \tilde{W}_t^*\tilde{\phi}_t(\tilde{\theta}_t^*(a)'\Sigma_t - \tilde{\Lambda}_t')dZ_t \tag{3.24}$$

Note that at this point we have not yet determined the optimal perturbation term a_t^* and thus the optimal strategy in the dual market is dependent on a_t

rather than a_t^*.

By equating the volatility terms from (3.23) and (3.24) we arrive at an equation that describes the optimal investment strategy in the dual market:

$$\left[\tilde{B}(t)\delta\rho - \tfrac{\gamma-1}{\gamma}\eta_1 \quad \tilde{B}(t)\delta\sqrt{1-\rho^2} - \tfrac{\gamma-1}{\gamma}\tilde{\eta}_{2,t}\right]\sqrt{\nu_t} = \tilde{\theta}_t^*(a)'\Sigma_t - \tilde{\Lambda}_t' \tag{3.25}$$

Solving the equation above for $\tilde{\theta}_t^*(a)'$ leads to the following optimal investment strategy in the dual market:

$$\begin{cases} \tilde{\theta}_{1,t}^*(a) = \dfrac{\eta_1}{\gamma} - \dfrac{(\eta_2 + a_t)\rho}{\gamma\sqrt{1-\rho^2}} - \theta_{2,t}^*(a)\dfrac{h_S S_t}{O_t} \\ \tilde{\theta}_{2,t}^*(a) = \left(\tilde{B}(t) + \dfrac{\eta_2 + a_t}{\gamma\delta\sqrt{1-\rho^2}}\right)\dfrac{O_t}{h_\nu} \end{cases} \tag{3.26}$$

Now that we have found the optimal strategy in the dual market it remains to be answered how we can link this to a strategy in the original incomplete market. Therefore, we determine a_t such that $\tilde{\theta}_{2,t}$ reduces to zero for every t. We can achieve this by choosing $a_t^* = -\tilde{B}(t)\gamma\delta\sqrt{1-\rho^2} - \eta_2$. Note that the optimal perturbation term depends on γ. Hence, every agent will choose a different perturbation term. Intuitively this can be explained by the fact that volatility risk is non-traded in the original market. Therefore, infinitely many pricing kernels exist in this market. Hence, how volatility risk is experienced, and thus how the dual strategy is made admissible in the primal market, is agent specific. Furthermore, the incompleteness is presented within the context of a primal-dual optimization problem for an agent with risk aversion parameter γ; the nature of the whole optimization problem is therefore agent dependent. Consequently, the way in which the evolution of dual wealth is linked to the evolution of primal wealth is also specific to each individual agent.

With the choice of a_t^* that reduces the demand of O_t to zero, we can present the system of ODE's in (3.21) that makes the dual strategy admissible in the primal market, for an agent with risk aversion parameter γ. Formally, $\dot{\tilde{A}}(t)$ and $\dot{\tilde{B}}(t)$ can be simplified as follows on the basis of a_t^*:

$$\begin{cases} \dot{\tilde{A}}(t) = -\kappa\bar{\nu}\tilde{B}(t) + \dfrac{\gamma-1}{\gamma}r \\ \dot{\tilde{B}}(t) = \Big(-\dfrac{1}{2}\delta^2 - \dfrac{1}{2}(\gamma-1)\delta^2(1-\rho^2)\Big)\tilde{B}(t)^2 + \Big(\kappa + \dfrac{\gamma-1}{\gamma}\delta\rho\eta_1\Big)\tilde{B}(t) \\ \qquad\quad + \dfrac{1}{2}\dfrac{\gamma-1}{\gamma^2}\eta_1^2 \end{cases} \tag{3.27}$$

where we again have the boundary conditions $\tilde{A}(T) = \tilde{B}(T) = 0$. We see that the price of risk of the second Brownian motion η_2 is absorbed by a_t as a consequence of duality theory. Hence, the dependency of the system on η_2 vanishes if the dual strategy is made admissible in the primal market.

The ODE describing $\tilde{B}(t)$ is a one dimensional Riccati equation with constant coefficients $a = -\frac{1}{2}\delta^2 - \frac{1}{2}(\gamma - 1)\delta^2(1 - \rho^2)$, $b = \kappa + \frac{\gamma-1}{\gamma}\delta\rho\eta_1$, and $c = \frac{1}{2}\frac{\gamma-1}{\gamma^2}\eta_1^2$. We therefore know that a closed-form solution exists. Nielsen and Jönsson (2015) show in Lemma 1 of their paper that $\tilde{B}(t)$ looks as follows:

$$\tilde{B}(t) = \begin{cases} \dfrac{2c(e^{D(t-T)} - 1)}{(D - b)(e^{D(t-T)} - 1) + 2D}, & \text{if } b^2 > 4ac, \\ \dfrac{2c(t - T)}{2 - b(t - T)}, & \text{if } b^2 = 4ac, \\ \dfrac{4ic\sin(\frac{\epsilon(t-T)}{2})}{2i\epsilon\cos(\frac{\epsilon(t-T)}{2}) - 2bi\sin(\frac{\epsilon(t-T)}{2})}, & \text{if } b^2 < 4ac. \end{cases} \tag{3.28}$$

with $D = \sqrt{b^2 - 4ac}$, $\epsilon = -iD$, and i the complex unit. Note that Nielsen and Jönsson (2015) give the solution for a problem with boundary condition $B(0) = 0$. Therefore the solution in (3.28) is slightly modified compared to the solution by Nielsen and Jönsson (2015) so that the terminal condition $B(T) = 0$ is met.

Combining all information leads to the optimal investment strategy in the original incomplete market, which we formally introduce in Proposition 3.1:

Proposition 3.1 *Consider the optimal dynamic investment problem in the Heston model specified in* (3.10)*. The corresponding optimal wealth process in the incomplete market,* W_t^**, is given by:*

$$W_t^* = \frac{W_0}{g_T}(\tilde{\phi}_t)^{-\frac{1}{\gamma}}\tilde{P}(t, \nu_t)$$

for all $t \in [0, T]$*. Here,* $g_T = \mathbb{E}\Big[(\tilde{\phi}_T)^{1-\frac{1}{\gamma}}\Big]$*,* $\tilde{\phi}_t$ *is defined in* (3.12)*, and* $\tilde{P}(t, \nu_t) = \exp\Big(\tilde{A}(t) + \tilde{B}(t)\nu_t\Big)$*.* $\tilde{\phi}_t$ *and* $\tilde{P}(t, \nu_t)$ *depend on the optimal perturbation term* a_t^**, which is set such that an agent does not invest in* O_t*:*

$$a_t^* = -\tilde{B}(t)\gamma\delta\sqrt{1 - \rho^2} - \eta_2$$

With the use of this perturbation term we have defined $\tilde{B}(t)$ *in* (3.28)*. Consequently,* $\tilde{A}(t)$ *looks as follows:*

$$\tilde{A}(t) = \int_0^t -\kappa\bar{\nu}\tilde{B}(s) + \frac{\gamma - 1}{\gamma}r ds$$

Ultimately, this leads to the following optimal fraction of wealth that is invested in the stock in the incomplete Heston market:

$$\theta_{1,t}^*(a^*) = \frac{\eta_1}{\gamma} + \tilde{B}(t)\delta\rho$$

The remainder $1 - \theta_{1,t}^*(a^*)$ *is invested in the money market account.*

In line with the mutual fund separation theorem introduced by Merton (1971), we see that the optimal demand of the stock consists of the mean-variance optimal demand and hedge demand. The first term shows that the mean-variance demand of the stock is proportionally related to the price of risk coefficient of the first Brownian motion. Hence, the mean-variance demand of the stock will increase as the price of risk of the stock increases. Furthermore, we see that the mean-variance demand of the stock is inversely related to the risk preference parameter, and thus the allocation to the stock will decrease if the risk aversion level increases. For a given risk aversion level, the mean-variance optimal demand is constant and thus independent of the realized stochastic volatility. This can be explained from the fact that we assumed that the prices of risk are proportionally related to the square root of the stochastic variance. Due to this assumption, the stochastic volatility in the denominator of the mean-variance demand has canceled out with the stochastic volatility in the numerator. Hence, the mean-variance demand is independent of the stochastic volatility. The hedge demand term can be explained by the fact that the evolution of the stock is influenced by the stochastic volatility (Liu and Pan, 2003). The two Brownian motions driving the uncertainty in the market are correlated with coefficient ρ. The stock only gives the possibility to take a position in the first Brownian motion, whereas the second Brownian motion indirectly influences the evolution of the stock via the stochastic volatility. The hedge demand corrects for this correlation effect.

We conclude this section by comparing the optimal strategy defined in Proposition 3.1 to the strategies presented in the papers we have discussed in the introduction to this chapter. Liu and Pan (2003), Chen et al. (2018), and Yang and Pelsser (2023) all find the demand of the stock to consist of three terms, similar to the structure we have found up to the point that we have set the demand of the added asset to zero. Hence, they present the optimal strategy in a complete market and thus the demand of the extra asset is not reduced to zero. In the dual market we find a similar demand for O_t as Liu and Pan (2003), Chen et al. (2018), and Yang and Pelsser (2023). The definition of $\tilde{B}(t)$ in (3.28) aligns with the definition of the $H(T-t)$ function in the work of Liu and Pan (2003), Chen et al. (2018), Yang and Pelsser (2023), although they find this function via a Laplace transform whereas we use the results of Duffie and Kan (1996). We argue that this constitutes technically less demanding derivations. The overall conclusion is that similar results are found as in Liu and Pan (2003), Chen et al. (2018), and Yang and Pelsser (2023), up to the definition of the optimal strategy in the incomplete market. We have thus extended the results in the literature by formally finding the optimal strategy in the incomplete market via duality theory. Although we argue that this is a methodological refinement, we conclude that the different methodologies lead to the same allocations in Heston's stochastic volatility model. Finally, Nielsen and Jönsson (2015) solve the optimal investment problem via a dynamic method and thus an optimal strategy for the incomplete market is presented. Nielsen and Jönsson (2015) find the demand of the stock to be a composition of mean-variance optimal demand and a correction term, which is in line with the results we present in Proposition 3.1.

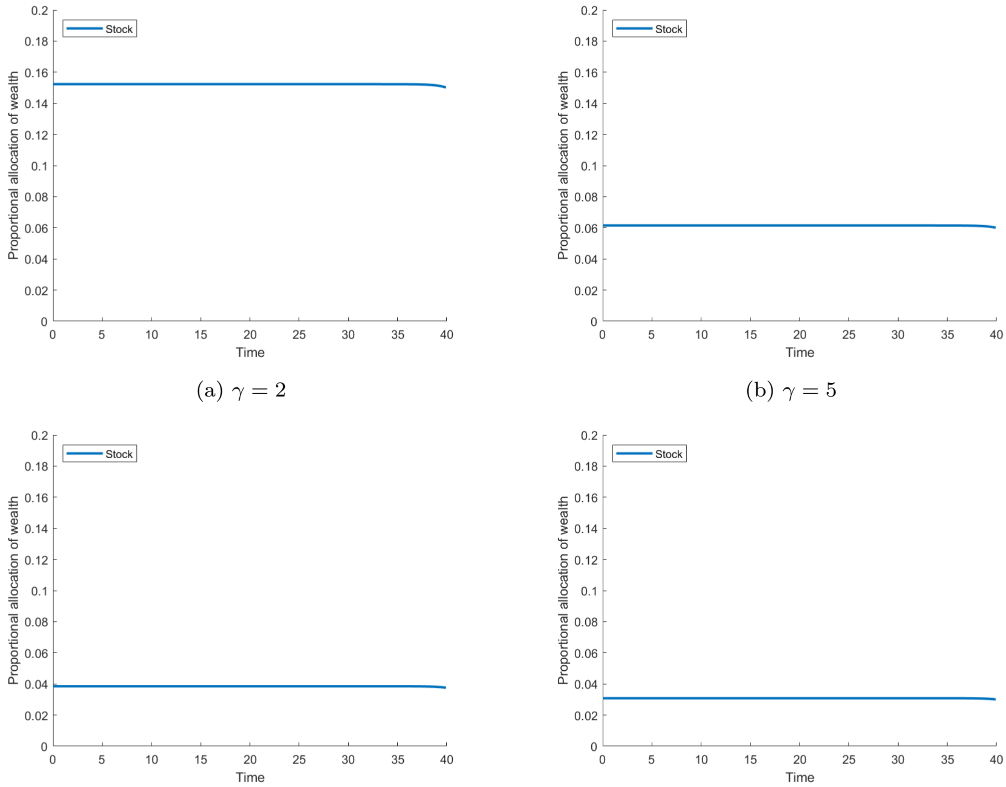
3.3 Numerical implementation investment strategy

In this section, we present the results of a numerical implementation of the investment strategy. As baseline input we take the parameter estimates used by Chen et al. (2018) and Yang and Pelsser (2023). The parameter estimates are presented in Table 3.1. Note that the estimates for the prices of risk of the Brownian motions below lead to a similar norm of the price of risk as is the case for baseline parameters in the KNW model, introduced in Table 2.1. Significantly different allocations to the stock can thus not be explained by significantly different coefficients of the price of risk.

Parameter	Value	Parameter	Value
ν_t		Λ_t, r	
ν_0	1.5	η_1	0.3
κ	1.0	r	2.4%
$\bar{\nu}$	1.5		
δ	0.5		
ρ	-0.4		

Table 3.1: Benchmark values for the Heston parameters, proposed by Chen et al. (2018) and Yang and Pelsser (2023).

In Figure 3.1 we present the (time dependent but not stochastic) optimal allocation to the stock over the whole investment horizon for different values of γ. The allocation to O_t is zero by construction and therefore excluded from the graphs. We simulate $m = 10,000$ scenarios with initial wealth $W_0 = 1$. However, the optimal strategy in the Heston model is constant across the scenarios and thus we do not show a percentile range around the allocation to the stock. The same random numbers are used for all simulations in this chapter and in the KNW chapter, for which the results are presented in Section 2.3 and Section 2.4. We have used an Euler scheme with monthly time steps in the simulations to simulate the stock and volatility process. To prevent the volatility process from becoming negative, we truncate the volatility at zero in our discretized SDE. Any remaining negative volatility values are set equal to the long-run mean $\bar{\nu}$. It should be noted that the volatility can be simulated with schemes that significantly reduce the bias. A well-known method is the quadratic exponential method introduced by Andersen (2007). Although this method leads to more accurate simulations of the stochastic volatility process, we expect that this does not influence the life cycles we examine in this thesis. The main reason for this is that we find little variation in simulated samples of the stochastic variance for different random numbers. This suggests that the results we find contain little variation. Furthermore, our simulation scheme ensures that the average value of the volatility is equal to the long-run mean of ν_t at each time point. We thus argue that by using enough scenarios and time steps we have mitigated the possible bias induced by the Euler scheme.



(a) $\gamma = 2$ (b) $\gamma = 5$

(c) $\gamma = 8$ (d) $\gamma = 10$

Figure 3.1: Optimal allocation to the stock in the Heston model. We assume $T = 40$ with monthly time steps in the simulations and $W_0 = 1$. Other parameter values are given in Table 3.1.

The fact that we find no variation in the optimal allocation to the stock can be explained completely from the optimal strategy in Proposition 3.1. The mean-variance optimal demand is constant over the time horizon, for a given risk aversion level, due to the proportionality assumption on the prices of risk. Hence, the only time variation in the allocation to the stock comes from the first hedge demand. However, as $B(t)$ is time dependent but not stochastic, the hedge demand is also constant across scenarios. Therefore, we find no variation in the optimal strategy across scenarios. Furthermore, we find that the magnitude of the first hedge demand is small compared to the mean-variance optimal demand. Since $B(t) < -0.02$ for each t and risk aversion level, $|\rho| < 1$ and $|\delta| < 1$, we find that the value of the second hedge demand is small compared to the value of $\frac{\eta_1}{\gamma}$. We thus conclude that the optimal strategy in the model of Heston (1993) is almost identical to the optimal strategy in the model of Black and Scholes (1973). In Table 3.1 we see that the correlation between Z_t^1 and Z_t^2 is negative and thus the agent implicitly takes a short position in the stochastic volatility via its position in the stock. To compensate for this short position, we can see in Figure 3.1 that the agent slightly reduces its position in the stock near the end of the time horizon. This happens because the agent then wants to lock in the terminal wealth and thus the risk that the short position in the volatility brings, is mitigated. On the other hand, if ρ would have been positive, one would see the opposite effect at the end of the horizon.

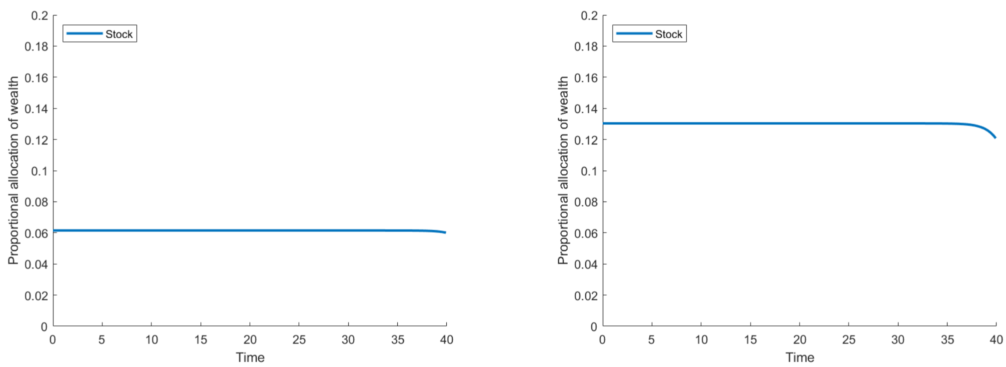
Another way of interpreting the small deviation of the optimal strategy in the stochastic volatility model from the optimal strategy in the Black-Scholes financial market, can be found in the setup of the stochastic variance process. From these dynamics we know that the mean value of the stochastic variance over the whole time horizon will approximately equal its long-run mean. If the volatility of the variance process is not too large, this process results in approximately constant values over the time horizon, in expectation. Therefore the volatility of the stock will in expectation not differ much from the volatility in the Black-Scholes market, if the constant volatility equals the long-run mean of the stochastic volatility process in Heston's model. Summing up, we find a strategy approximately similar to Merton's mean-variance optimal strategy.

Liu and Pan (2003) argue that the optimal strategy in Heston's model is sensitive with respect to the price of risk of the first Brownian motion. As we have shown above, the optimal strategy is almost identical to the mean-variance optimal strategy and thus scaling η_1 will influence the optimal investment strategy equally. We argue that also the volatility of the stochastic variance process influences the optimal strategy. The higher the volatility of the variance, the more Heston's financial market will differ from the Black-Scholes financial market. Consequently, the optimal strategy will also deviate more from the mean-variance optimal strategy. To investigate the sensitivity of the allocations with resect to the parameter estimates we present an alternative set of parameters for which the numerical allocations are presented. In this set we double the price of risk of the first Brownian motion and increase the volatility of the stochastic variance, as can be seen in Table 3.2. The model contains several other parameters for which a sensitivity analysis could be performed. However, an individual analysis per parameter would be too lengthy. The choice to only investigate the sensitivity with respect to η_1 and δ is motivated by the work of Liu and Pan (2003) and the argument that Heston's model will differ more from the Black-Scholes market for higher values of δ.

Parameter	Value	Parameter	Value
ν_t		Λ_t, r	
ν_0	1.5	η_1	0.6
κ	1.0	r	2.4%
$\bar{\nu}$	1.5		
δ	0.8		
ρ	-0.4		

Table 3.2: Alternative values for the Heston parameters, proposed by Chen et al. (2018) and Yang and Pelsser (2023).

In Figure 3.2 we compare the strategy following from the baseline and alternative parameters for $\gamma = 5$. Compared to the baseline parameters, we find a more aggressive strategy. Doubling the price of risk of the first Brownian motion will double the mean-variance optimal demand. However, the strategy in Figure 3.2b is not exactly two times the strategy in Figure 3.2a. Any other effect can thus be explained by the hedge demand. Although the hedge demand is also influenced by η_1, the difference in this term can mostly be explained by the increased volatility of the variance process. Due to this increased volatility, the effect of the short position in the stochastic volatility on the optimal portfolio has increased. Therefore, the agent faces more volatility risk by holding a position in the stock and thus it de-risks more extremely to lock in on the terminal wealth. In Figure B.1 we present the optimal strategies for the alternative parameters for $\gamma = 2$, $\gamma = 8$ and $\gamma = 10$. For all levels of γ we see a similar pattern where the strategy gets almost twice as aggressive as compared to the baseline parameters. Earlier de-risking can again be explained by the increased volatility.



(a) Baseline parameters, $\gamma = 5$

(b) Alternative parameters, $\gamma = 5$

Figure 3.2: Comparison of the optimal strategy in the Heston model for the baseline and alternative parameters, for $\gamma = 5$. We assume $T = 40$ with monthly time steps in the simulations and $W_0 = 1$. The baseline parameters are given in Table 3.1, the alternative parameters in Table 3.2.

Overall we conclude that the optimal strategy in Heston's stochastic volatility model is almost identical to the optimal strategy in the Black-Scholes financial market. Furthermore, the optimal strategy is highly sensitive with respect to the parameter estimates. Although the proportional allocations are not as extreme as in the KNW model, we see that small deviations in the parameters can lead to large differences in the optimal strategy. Therefore, if the Heston model is used in the pension-industry, parameter uncertainty will play an important role. However, since the Heston model leads to seemingly similar investment strategies as the Black-Scholes model, one could question whether a more sophisticated model is needed in practice. In the next section we will analyze the welfare effects of investing according to Merton's myopic strategy in the model of Heston. On the basis of these results, we can argue whether taking volatility risk into account will benefit the participants of a pension fund.

3.4 Welfare analysis

In this section we will compare the welfare generated by the optimal investment strategy in the Heston model with the welfare generated by the mean-variance optimal strategy found by Merton (1969). Furthermore we will investigate the welfare effects of using an incorrect risk preference parameter. Although the mean-variance optimal strategy in Heston's stochastic volatility model logically follows from Proposition 3.1, we formally define it below:

Definition 3.1 *Consider the optimal investment strategy in the Heston model specified in Proposition 3.1. We define the corresponding myopic strategy as:*

$$\theta_{1,t} = \frac{\eta_1}{\gamma}$$

where η_1 is the constant price of risk of the first Brownian motion.

Note that the structure of the KNW model needed us to define two different mean-variance strategies. However, in the model of Heston (1993), the myopic strategy is already a static strategy and thus no additional strategy is defined.

In Section 3.3 we have found that the optimal strategy in Heston's model is close to the myopic strategy. Therefore, we expect that investing according to the myopic strategy will lead to small welfare losses, specifically in comparison to the welfare losses we found in Section 2.4 for the KNW model. In Table 3.3 we present the percentage welfare losses when using the suboptimal myopic strategy, instead of the optimal strategy for both the baseline and alternative parameters. The underlying certainty equivalents and their standard errors can be found in Table B.1.

γ	2	3	4	5	6	7	8	9	10
Myopic baseline (%)	-0.06	-0.06	-0.04	-0.03	-0.02	-0.01	-0.01	-0.01	-0.01
	(0.06	(0.10)	(0.12)	(0.12)	(0.11)	(0.11)	(0.10)	(0.10)	(0.09)
Myopic alternative (%)	-0.76	-3.17	-4.45	-4.88	-4.92	-4.80	-4.60	-4.38	-4.17
	(1.36)	(1.47)	(1.23)	(1.05)	(0.91)	(0.80)	(0.71)	(0.64)	(0.58)

Table 3.3: Rounded percentage loss in certainty equivalent when using the myopic strategy defined in Definition 3.1 instead of the optimal strategy defined in Proposition 3.1, for the baseline and alternative parameters. The baseline parameters can be found in Table 3.1, the alternative parameters in Table 3.2. Percentage standard errors are reported in parentheses. The data is calculated over $10,000$ simulations. We assume $T = 40$ with monthly time steps in the simulations and $W_0 = 1$.

For the baseline parameters we find no welfare losses statistically significant different from zero and thus we cannot conclude a difference in certainty equivalents between the optimal strategy and the myopic strategy. The equivalency of the optimal strategy with the mean-variance optimal strategy we concluded in Section 3.3 is thus supported by the findings above, for the baseline parameters. For the alternative parameters we can conclude statistically significant welfare losses for all values of γ different than two. This is primarily motivated by the fact that the volatility of the stochastic variance has increased in the alternative parameter set. Due to this increased volatility, the relative allocation of the hedge demand compared to the mean-variance optimal demand increases. Hence, investing myopically will lead to higher welfare losses. For $\gamma \geq 3$ we cannot conclude a statistically significant difference between the welfare losses. Hence, investing myopically will affect each investor approximately similar, independent of the risk aversion level. The fact that we do not find a statistically significant welfare loss for $\gamma = 2$ can be explained from the relative importance of the hedge portfolio, as compared to the mean-variance optimal portfolio. For $\gamma = 2$ the mean-variance optimal portfolio relatively has the highest share in the total optimal portfolio, compared to the other values of the risk aversion parameter. Hence, excluding the hedge portfolio will affect this agent the least. We see that for $\gamma \geq 3$ the relative share of the hedge portfolio is approximately the same for each risk aversion level.

Although the alternative parameters lead to statistically significant welfare losses, the welfare losses are of a significantly smaller magnitude than the welfare losses of a myopic investor in the KNW model. Hence, not taking volatility risk in Heston's model into account affects an agent significantly less than not taking the interest rate risk and inflation risk in the model of Koijen et al. (2010) into account. Hence, if the KNW model is assumed to represent the economy, we state that investing according to the Black-Scholes model will significantly hurt the agent. However, for the Heston model this is less clear. One could argue that the extra complexity caused by the stochastic volatility does not lead to fundamentally different optimal strategies and thus using the Black-Scholes model is instead is more favorable. On the other hand, we find that, for certain parameter choices, investing myopically will lead to welfare losses statistically significant different from zero. We thus argue that it remains to be an open discussion whether or not taking volatility risk into account is useful. It should be noted that the parameter sensitivity persists when using the Black-Scholes model instead of the stochastic volatility model. Also in the Black-Scholes market the mean-variance optimal demand is highly sensitivity with respect to the price of risk of the Brownian motion that drives the stock. Hence, the core of discussion about using Heston's model or the Black-Scholes model lies in the question whether the extra complexity outweighs the possible welfare losses.

We conclude this section by presenting the welfare losses from using an incorrect risk preference parameter. We present the welfare losses of using an incorrect risk preference parameter for the baseline and alternative parameters in Table 3.4 and Table 3.4, respectively. The corresponding certainty equivalents and their standard errors can be found in Table B.2 and Table B.3, respectively. The tables with welfare losses present the welfare losses if an agent invests according to the γ_{measured} on the horizontal axis, whereas the strategy is evaluated on the basis of the γ_{true} on the vertical axis. Further information on how the certainty equivalents and their standard errors are calculated can be found in Section 1.3.

True γ \ Measured γ	2	3	4	5	6	7	8	9	10
2	–	-14.85	-29.93	-39.89	-46.55	-51.22	-54.63	-57.23	-59.26
		(0.82)	(0.87)	(0.83)	(0.78)	(0.74)	(0.70)	(0.67)	(0.65)
3	-22.23	–	-6.04	-14.47	-21.49	-26.97	-31.24	-34.63	-37.36
	(3.69)		(0.90)	(1.09)	(1.11)	(1.09)	(1.06)	(1.03)	(1.00)
4	-52.25	-8.37	–	-2.91	-7.85	-12.57	-16.65	-20.07	-22.94
	(4.94)	(2.37)		(0.79)	(1.05)	(1.14)	(1.16)	(1.15)	(1.14)
5	-68.20	-23.93	-4.01	–	-1.58	-4.65	-7.89	-10.89	-13.55
	(3.63)	(4.16)	(1.57)		(0.66)	(0.94)	(1.05)	(1.10)	(1.12)
6	-75.99	-37.21	-12.35	-2.25	–	-0.93	-2.95	-5.24	-7.47
	(2.57)	(4.34)	(2.96)	(1.10)		(0.54)	(0.80)	(0.94)	(1.00)
7	-80.26	-46.45	-21.05	-7.14	-1.39	–	-0.58	-1.97	-3.63
	(1.92)	(3.86)	(3.59)	(2.13)	(0.81)		(0.44)	(0.68)	(0.82)
8	-82.89	-52.75	-28.37	-12.80	-4.49	-0.93	–	-0.38	-1.37
	(1.50)	(3.32)	(3.64)	(2.79)	(1.58)	(0.62)		(0.37)	(0.59)
9	-84.65	-57.17	-34.11	-18.14	-8.31	-3.02	-0.66	–	-0.26
	(1.22)	(2.86)	(3.45)	(3.07)	(2.16)	(1.22)	(0.49)		(0.31)
10	-85.90	-60.40	-38.56	-22.76	-12.18	-5.69	-2.13	-0.48	–
	(1.02)	(2.49)	(3.17)	(3.09)	(2.50)	(1.71)	(0.96)	(0.39)	

Table 3.4: Rounded percentage loss in certainty equivalent when the agent invests according to an incorrect risk aversion parameter instead of the true risk preference parameter. Percentage standard errors are reported in parentheses. The data is calculated over 10,000 simulations. We assume $T = 40$ with monthly time steps and $W_0 = 1$. Other parameter values are given in Table 3.1.

True γ \ Measured γ	2	3	4	5	6	7	8	9	10
2	–	-44.53	-74.63	-86.44	-91.64	-94.28	-95.77	-96.69	-97.30
		(3.91)	(2.18)	(1.23)	(0.78)	(0.54)	(0.40)	(0.31)	(0.26)
3	-39.29	–	-22.94	-47.12	-62.74	-72.37	-78.50	-82.59	-85.44
	(6.16)		(4.80)	(4.67)	(3.69)	(2.87)	(2.29)	(1.88)	(1.58)
4	-54.13	-2.68	–	-15.01	-31.42	-44.65	-54.52	-61.78	-67.19
	(4.63)	(5.60)		(3.69)	(4.74)	(4.58)	(4.10)	(3.60)	(3.17)
5	-58.22	-6.43	5.56	–	-11.02	-22.60	-32.88	-41.41	-48.30
	(3.77)	(7.00)	(3.96)		(2.72)	(4.03)	(4.42)	(4.36)	(4.11)
6	-59.35	-7.62	7.75	6.76	–	-8.54	-17.18	-25.16	-32.19
	(3.28)	(7.17)	(5.78)	(2.79)		(2.04)	(3.28)	(3.90)	(4.11)
7	-59.46	-7.52	9.48	10.71	6.31	–	-6.84	-13.58	-19.90
	(2.94)	(6.89)	(6.47)	(4.37)	(2.04)		(1.57)	(2.66)	(3.33)
8	-59.19	-6.84	11.14	13.58	10.50	5.54	–	-5.60	-11.03
	(2.67)	(6.44)	(6.58)	(5.15)	(3.32)	(1.54)		(1.24)	(2.17)
9	-58.75	-5.92	12.73	15.91	13.62	9.47	4.78	–	-4.67
	(2.45)	(5.94)	(6.40)	(5.46)	(4.05)	(2.57)	(1.20)		(1.00)
10	-58.26	-4.88	14.25	17.92	16.12	12.49	8.33	4.11	–
	(2.25)	(5.45)	(6.09)	(5.49)	(4.42)	(3.22)	(2.04)	(0.96)	

Table 3.5: Rounded percentage loss in certainty equivalent when the agent invests according to an incorrect risk aversion parameter instead of the true risk preference parameter. Percentage standard errors are reported in parentheses. The data is calculated over $10,000$ simulations. We assume $T = 40$ with monthly time steps and $W_0 = 1$. Other parameter values are given in Table 3.2.

In contrast to the welfare losses based on the myopic strategy we find that using an incorrect risk aversion parameter can lead to statistically significant welfare losses. For the baseline parameters we find that the welfare losses increase as the measured value of γ lies further away from the true value. Overall we find a similar pattern as we found when using an incorrect risk aversion parameter in the KNW model: absolute deviations from the true risk aversion parameter of at most one will in general lead to relatively small welfare losses, especially if the true risk aversion parameter increases. This can be explained by the fact that the optimal strategy in Heston's model is almost identical to the mean-variance optimal strategy. Hence, the mean-variance allocation will converge to zero if γ_{true} converges to infinity. Consequently, a small misestimate will have larger impact on the corresponding portfolio, for the lowest values of the true risk preference parameter. Furthermore, we see that an underestimation of the risk preference parameter in general leads to larger welfare losses than an overestimation. This can be explained by the fact that underestimations lead to relatively higher convergence from the true mean-variance demand than overestimations. Hence, welfare losses will increase more rapidly in these cases.

For the alternative parameters it should be noted that we find some positive percentages. Although a suboptimal strategy can theoretically never result in a welfare gain, positive numbers can be found as a consequence of the numerical implementation. The optimal strategies in Heston's model are extremely close to each other, especially if γ_{true} increases. Hence, the certainty equivalent of the suboptimal strategy can become larger than the certainty equivalent of the optimal strategy, due to numerical errors. In most of these case the certainty equivalents overlap when taking the standard errors into account. Hence, no statistically significant difference between the welfare losses can be found, and thus the two strategies can be assumed to be identical. In these cases it might happen that the welfare loss itself is not significantly different from zero, taking standard errors into account. This can also be explained by numerical errors. The welfare losses itself are a non-linear transformation of the certainty equivalents. Furthermore, the method we use to calculate the standard errors of the welfare loss provides an approximation of the true standard errors. Hence, if two strategies are extremely close to each other, we might find welfare losses that are different from zero although the certainty equivalents are not different from each other. It should be noted that for the alternative parameters only positive welfare losses are found if both the true and measured risk aversion parameter are high. In these cases the strategies are so close to each other that it is more likely that numerical noise will lead to wrong outcomes. Furthermore, if the strategies are so close to each other, it might happen that investing a bit more aggressive than should be done according to the true risk aversion parameter will benefit the agent. This can be explained by the non-linear relationship between the simulated terminal wealth and the calculated values of the certainty equivalents. Since we perform an ex-post evaluation of the simulated wealth, evaluating the strategies following from, for example, $\gamma_{\text{true}} = 8$ and $\gamma_{\text{measured}} = 5$ on the basis of γ_{true} might result in a higher certainty equivalent for the strategy simulated according $\gamma = 5$, due to the fact that the strategies are so close to each other. Overall we conclude that numerical noise can arise due the fact that the strategies are close to each other, especially for high values of γ, and all non-linear relations underlying the certainty equivalents.

It should be noted that we do not find positive welfare losses for the baseline parameters. Hence, we can conclude that the optimal strategies are relatively more close to each other across different values of γ for the alternative parameters than for the baseline parameters. A possible explanation for this can be the different values for the volatility of the stochastic variance between the two parameter sets. As a consequence of this, the relative importance of the hedge portfolio in contrast to the mean-variance optimal portfolio will change, for a given value of γ. Furthermore, due to the increased volatility in the alternative parameter set, our numerical implementation of the stochastic volatility is truncated at zero more often. This can possibly lead to more noise around the welfare losses presented for the alternative parameters. It should be noted that we find similar results for the alternative parameters as for the baseline parameters for most values of γ_{true}. Welfare losses increase as the absolute difference between the true and measure risk aversion parameter increase and underestimations will generally hurt the agent more than overestimations.

We conclude this section by stating that we do not find similar welfare results for an agent that invests myopically across the KNW and Heston model. In the KNW model investing myopically will lead to significantly higher welfare losses than in the Heston model. However, investing according to an inaccurate risk preference parameter will lead to similar welfare losses in the two models. We thus state that not taking interest rate risk and inflation risk into account has more consequences than not taking volatility risk into account. On the other hand, the effects of assuming a wrong risk preference parameter is approximately constant across the models.

Chapter 4

CP2022

In 2022 the Commissie Parameters advised the Dutch government that the KNW financial market model does not meet the requirements set by the new Dutch pension law (Ministerie van Sociale Zaken en Werkgelegenheid, 2022a). Therefore, the commission proposed a new model to generate the scenario sets used by pension funds, the CP2022 model. The model provides a better match with the market prices of risk observed in the real world. Furthermore, it can fit the nominal and real term structure more accurately (Ministerie van Sociale Zaken en Werkgelegenheid, 2022a). The CP2022 model builds on the framework of Koijen et al. (2010) by providing term structures that are affine in the state variables. Furthermore, it incorporates stochastic volatility in the model. The model by Commissie Parameters (2022) can thus be seen as a combination of the models by Koijen et al. (2010) and Heston (1993). Under the new Dutch pension law, the use of the CP2022 model is mandatory for the analysis of future financial situations of pension funds (Ministerie van Sociale Zaken en Werkgelegenheid, 2022b). However, little information on the implications of the model is publicly available. For example, no numerical implementation of the model is published by DNB and no research on optimal investment in the CP2022 model is available.

Therefore we will provide an analysis of the CP2022 model in the context of optimal investment. In Section 4.1 we will present the financial market. Similar to the model by Koijen et al. (2010) and Heston (1993), the original market is incomplete. However, the CP2022 model contains two untraded sources of risk: realized inflation risk and volatility risk. Therefore, we will present the dynamics of an inflation-indexed bond and an asset that trades the source of risk underlying the stochastic volatility. With the extra assets we arrive at a complete market setup. In Section 4.2 we will show that the model by Commissie Parameters (2022) allows us to find an analytical expression for the optimal wealth. However, no analytical optimal strategy can be found on the basis of currently known techniques. Therefore, it remains to be answered what the optimal investment strategy in the CP2022 is and what the effects of neglecting the fundamental risks in the model are for the Dutch pension participants. In Section 4.3 we will show that modifications of the model can lead to alternative, possibly near-optimal, analytical investment strategies. However, numerical implementation of the alternative strategies turns out to be complex due to the analytical structure of the model. Therefore, we cannot present the welfare losses of investing according to this alternative strategy. We choose to present these results as they provide a basis for further research on optimal investment in the CP2022 model.

There is no information available in the literature on solving Merton's portfolio problem in the model of Commissie Parameters (2022). However, optimal investment in a stochastic interest rate and volatility model has been studied in alternative contexts. Noh and Kim (2011) explore the optimal strategy over an infinite horizon in a framework where the interest rate and volatility are driven by a Markov process. However, this constitutes a fundamental difference from the framework in which bond prices are assumed to be exponentially affine in the state variables. Furthermore, Noh and Kim (2011) do not account for stochastic inflation. Chang and Li (2016) provide a framework with affine short rates. However, also here stochastic inflation is not taken into account. Overall, we conclude that little information is available on optimal investment in an affine short rate model with stochastic volatility and inflation. Consequently, there is also no information on welfare analysis in such models.

4.1 Financial market

Commissie Parameters (2022) postulates an affine model in which the term structure is driven by the state vector X_t^s, similar to how Koijen et al. (2010) defined their model. In the KNW model, the nominal short rate and inflation are driven by the state vector. The CP2022 model is characterized by a modified setup where the nominal short rate and inflation are part of the state vector. The third state variable is the volatility, which is assumed to be stochastic. Furthermore, Commissie Parameters (2022) postulates a stochastic process for the natural logarithm of the stock and customer price index in the process X_t^o, whereas Koijen et al. (2010) models the stock and CPI dynamics directly, without the natural logarithm. The processes X_t^s and X_t^o can be combined in the vector X_t, leading to the following vectors:

$$X_t = \begin{bmatrix} X_t^s \\ X_t^o \end{bmatrix}, \quad X_t^s = \begin{bmatrix} \nu_t \\ r_t \\ \pi_t \end{bmatrix}, \quad X_t^o = \begin{bmatrix} \ln(S_t) \\ \ln(\Pi_t) \end{bmatrix} \tag{4.1}$$

where we thus have $X_t^s \in \mathbb{R}^{3\times 1}$, $X_t^o \in \mathbb{R}^{2\times 1}$, and $X_t \in \mathbb{R}^{5\times 1}$. The starting values $\nu_0 > 0$, r_0, and π_0 are provided as estimates in the scenario sets published by DNB. Furthermore, we assume $S_0 = \Pi_0 = 1$ so that $\ln(S_t) = \ln(\Pi_t) = 0$.

Commissie Parameters (2022) postulates the following dynamics for X_t^s:

$$dX_t^s = \begin{bmatrix} K_{\nu\nu} & 0 & 0 \\ K_{\nu r} & K_{rr} & K_{\pi r} \\ K_{\nu\pi} & K_{r\pi} & K_{\pi\pi} \end{bmatrix} \left(\begin{bmatrix} \mathbb{E}\nu_\infty \\ \mathbb{E}r_\infty \\ \mathbb{E}\pi_\infty \end{bmatrix} - X_t^s \right) dt + \begin{bmatrix} \omega & 0 & 0 & 0 & 0 \\ \sigma_{\nu r} & \sigma_{r1} & \sigma_{r2} & 0 & 0 \\ \sigma_{\nu\pi} & \sigma_{\pi 1} & \sigma_{\pi 2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \nu_t & 0_{1\times 4} \\ 0_{4\times 1} & I_4 + \nu_t \Gamma_1 \end{bmatrix}^{\frac{1}{2}} dZ_t$$
$$= K(\mathbb{E}X_\infty^s - X_t^s)dt + \Sigma^{r\pi}(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}} dZ_t \tag{4.2}$$

Here $Z_t \in \mathbb{R}^{5\times 1}$ is a 5-dimensional vector with independent Brownian motions and thus $\Sigma^{r\pi}(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}$ is the volatility of the state variables, which can be split up in two parts. $(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}$ can be interpreted as the multidimensional extension of $\sqrt{\nu_t}$ in the model of Heston (1993). Hence, it models the stochastic volatility of the vector X_t. Consequently, $\Sigma^{r\pi}$ can be interpreted as the volatility of the stochastic volatility. It determines how the stochastic volatility matrix influences the dynamics of X_t^s. We see that X_t^s is mean-reverting around its long-run mean $\mathbb{E}X_\infty^s \in \mathbb{R}^{3\times 1}$, with mean-reversion speed $K \in \mathbb{R}^{3\times 3}$. Consistent with Commissie Parameters (2022), we impose (i) $\omega > 0$, (ii) K and Γ_1 have real positive eigenvalues, (iii) Γ_1 has zero values outside its diagonal, and (iv) $K_{\nu\nu}\mathbb{E}\nu_\infty - \frac{1}{2}\omega^2 \geq 0$[18]. Note that the dynamics of X_t^s above impose the following structure of Γ_0 and Γ:

$$\Gamma_0 = \begin{bmatrix} 0 & 0_{1\times 4} \\ 0_{4\times 1} & I_{4\times 4} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0_{1\times 4} \\ 0_{4\times 1} & \Gamma_1 \end{bmatrix} \tag{4.3}$$

where $I_{4\times 4}$ is the 4-dimensional identity matrix. The stochastic volatility matrix is thus assumed to have an affine structure where Γ_0 represents the constant coefficient and Γ determines how the stochastic volatility enters the stochastic volatility matrix. Finally, it should be noted that Γ_1 is estimated as part of the model and thus DNB presents its estimated elements in the quarterly published scenario sets.

The dynamics of the logarithm of the stock and CPI are as follows:

$$dX_t^o = \begin{bmatrix} r_t + \eta_S \\ \pi_t + \eta_\pi \end{bmatrix} dt - \frac{1}{2} D\left(\begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix} \begin{bmatrix} \nu_t & 0_{1\times 4} \\ 0_{4\times 1} & I_4 + \nu_t\Gamma_1 \end{bmatrix} \begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix}' \right) dt + \begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix} \begin{bmatrix} \nu_t & 0_{1\times 4} \\ 0_{4\times 1} & I_4 + \nu_t\Gamma_1 \end{bmatrix}^{\frac{1}{2}} dZ_t$$
$$= (\mu^o + K^o X_t^s)dt + \Sigma^{S\Pi}(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}} dZ_t \tag{4.4}$$

where $\eta_S \in \mathbb{R}$, $\eta_\Pi \in \mathbb{R}$, $\sigma_S \in \mathbb{R}^{5\times 1}$, and $\sigma_\Pi \in \mathbb{R}^{5\times 1}$. Here $D(A)$ represents the diagonal of the matrix A as a column vector (Commissie Parameters, 2022). Simultaneously with defining the dynamics of X_t^o we have thus defined the

[18] The assumptions $K_{\nu\nu}\mathbb{E}\nu_\infty - \frac{1}{2}\omega^2 \geq 0$ and $\nu_0 > 0$ imply that $\mathbb{P}(\nu_t > 0) = 1$.

dynamics of μ^o and K^o:

$$\begin{aligned}\mu^o &= \begin{bmatrix}\eta_S\\ \eta_\pi\end{bmatrix} - \frac{1}{2}D\left(\Sigma^{S\Pi}\Gamma_0\Sigma^{S\Pi\prime}\right)\\ K^o &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \frac{1}{2}D\left(\Sigma^{S\Pi}\Gamma\Sigma^{S\Pi\prime}\right)\begin{bmatrix}1 & 0 & 0\end{bmatrix}\end{aligned} \tag{4.5}$$

The goal of μ^o and K^o is therefore to express the drift term of X_t^o in two separate parts: a part influenced by the state variables and a part not influenced by the state variables. In line with Commissie Parameters (2022) we assume $\sigma_{\Pi,4} = 0$, which is necessary to make the model specification unique. Note that a similar assumption is needed in the model of Koijen et al. (2010). The CPI defined in (4.4) links nominal quantities in this economy to real quantities and thus it fulfills a similar role as the CPI in the KNW model. In the original market investments can only be made in nominal assets and thus agents can only hedge themselves against fluctuations in the nominal economy, which partly explains the incompleteness of the market. We extend the original market with an inflation-indexed bond, so that all random sources influencing the evolution of $\ln \Pi_t$ can be traded. In combination with the introduction of an asset that trades the volatility risk, this will lead to a market with as many assets as risk drivers. In the dual market we lift the trading constraints on the inflation-indexed bond and volatility linked asset so that all sources of risk are traded. Finally, it should be noted that (4.4) only prescribes the dynamics of the natural logarithm of the stock and CPI. To facilitate further derivations we choose to also present the dynamics of S_t and Π_t (for derivations see Appendix C.1):

$$\begin{aligned}dS_t &= S_t(r_t + \eta_S)dt + S_t\sigma_S{}'(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t\\ d\Pi_t &= \Pi_t(\pi_t + \eta_\pi)dt + \Pi_t\sigma_\Pi{}'(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t\end{aligned} \tag{4.6}$$

We conclude the introduction of the state variables by presenting the dynamics of the complete vector X_t, which will facilitate further derivations:

$$dX_t = (M + LX_t)dt + \Sigma(G_0 + \sum_i X_t^i G_i)^{\frac{1}{2}}dZ_t \tag{4.7}$$

where have defined the following matrices:

$$M = \begin{bmatrix} K\mathbb{E}X_\infty^s \\ \mu^o \end{bmatrix}, \quad L = \begin{bmatrix} -K & 0_{3\times 2} \\ K^o & 0_{2\times 2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma^{r\pi} \\ \Sigma^{S\Pi} \end{bmatrix} \tag{4.8}$$

and $G_0 = \Gamma_0$, $G_1 = \Gamma$ and $G_i = 0_{5\times 5}$ for $i > 0$. It should be noted that the reformulation allows the volatility matrix to be written as a function of all variables in X_t rather than only as a function of ν_t. This will be useful when defining the bond price dynamics.

Commissie Parameters (2022) defines the market price of risk to be affine in X_t^s, up to division by the stochastic volatility matrix:

$$\begin{aligned}\Lambda_t &= ((\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}})^{-1}(\Lambda_0 + \Lambda_1 X_t^s) \\ &= ((G_0 + \sum_i X_t^i G_i)^{\frac{1}{2}})^{-1}(\Lambda_0 + \bar{\Lambda}_1 X_t)\end{aligned} \tag{4.9}$$

where $\Lambda_0 \in \mathbb{R}^{5\times1}$, $\Lambda_1 \in \mathbb{R}^{5\times3}$, and $\bar{\Lambda}_1 = [\Lambda_1 \quad 0_{5\times2}]$. We thus can write the price of risk as a function of X_t^s or as a function of X_t. Depending on the nature of the derivations we will shift between both forms[19]. The following constraints are imposed on Λ_0 and Λ_1:

$$\begin{aligned}\Sigma^{S\Pi}\Lambda_0 &= \begin{bmatrix}\eta_S \\ \eta_\pi\end{bmatrix} \\ \Sigma^{S\Pi}\Lambda_1 &= 0_{2\times3} \\ K^{\mathbb{Q}} &= K + \Sigma^{r\pi}\Lambda_1 \\ \Sigma^{r\pi}\Lambda_0 &= -K^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}X_\infty^s + K\mathbb{E}X_\infty^s \\ \Lambda_{1,(1,2)} &= 0 \\ \Lambda_{1,(1,3)} &= 0\end{aligned} \tag{4.10}$$

where $K^{\mathbb{Q}} \in \mathbb{R}^{3\times3}$ is the mean-reversion speed of X_t^s under the risk neutral measure, i.e. the measure described by the Radon-Nikodym derivative that is driven by Λ_t. $\mathbb{E}^{\mathbb{Q}}X_\infty^s \in \mathbb{R}^{3\times1}$ is defined as the long-run mean of X_t^s under the same risk neutral measure. Estimates for $K^{\mathbb{Q}}$ and $\mathbb{E}^{\mathbb{Q}}X_\infty^s$ will be given in the scenario sets published by the DNB. Hence, all elements of Λ_0 and Λ_1 can be uniquely determined using the constraints in (4.10).

Next to the stock, the agent can invest in a nominal money market account, paying the nominal short rate r_t. The dynamics of this money market account are as follows:

$$\frac{dB_t}{B_t} = r_t dt \tag{4.11}$$

where $B_0 = 1$. By virtue of using the money market account as numéraire, we can define the *nominal* pricing kernel:

$$\frac{d\phi_t^N}{\phi_t^N} = -r_t dt - \Lambda_t' dZ_t \tag{4.12}$$

with $\phi_0^N = 1$. We can use the nominal pricing kernel to define the real pricing kernel as $\phi_t^R = \phi_t^N \Pi_t$. In Appendix C.1 we find that ϕ_t^R adheres to the following

[19]In Chapter 3 we defined the prices of risk to be proportionally related to the stochastic volatility. However, the price of risk dynamics in the CP2022 model do not follow this structure. For this reason it turns out in Section 4.2 that no analytical investment strategy can be found.

dynamics:

$$\begin{aligned}\frac{d\phi_t^R}{\phi_t^R} &= -(r_t - \pi_t - \eta_\pi + \sigma_\Pi{}'(\Lambda_0 + \Lambda_1 X_t^s))dt - (\Lambda_t' - \sigma_\Pi{}'(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}})dZ_t \\ &= -R_t dt - \Lambda_t^{R\prime} dZ_t\end{aligned} \tag{4.13}$$

where $\phi_0^R = 1$. We define R_t as the instantaneous real short rate and $\Lambda_t^R = \Lambda_t - (\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}\prime}\sigma_\Pi$ as the real pricing kernel.

Next to the money market account and the stock, the agent can invest in two nominal bonds with different maturities. To find the price dynamics of the nominal bonds, we rely on the work of Duffie and Kan (1996). Since the short rate is described by a parametric Markov diffusion process, the results of Duffie and Kan (1996) can be utilized. Therefore we know that the price of a nominal bond maturing at time $t + \tau_i$ is an exponentially affine function of the state variables. This price is denoted by $P_{t+\tau_i}^N(t, X_t^s) = \mathbb{E}\left[\frac{\phi_{t+\tau_i}^N}{\phi_t^N}\middle|\mathcal{F}_t\right] = \exp\left(A^N(\tau_i) + B^N(\tau_i)' X_t^s\right)$ for $i = 1, 2$ with $\tau_1 \neq \tau_2$. By exploiting the martingale property of $\phi_t^N P_{t+\tau_i}^N(t, X_t^s)$ we find the following nominal bond price dynamics (see Appendix C.1 for derivations):

$$\frac{dP_{t+\tau_i}^N(t,X_t^s)}{P_{t+\tau_i}^N(t,X_t^s)} = \left(r_t + B^N(\tau_i)'\Sigma^{r\pi}(\Lambda_0 + \Lambda_1 X_t^s)\right)dt + B^N(\tau_i)'\Sigma^{r\pi}(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}} dZ_t \tag{4.14}$$

The functions $A^N(\tau_i) \in \mathbb{R}$ and $B^N(\tau_i) \in \mathbb{R}^{3\times 1}$ are defined according to the following ODE's:

$$\begin{cases}\dot{A}^N(\tau_i) = B^N(\tau_i)'(K\mathbb{E}X_\infty^s - \Sigma^{r\pi}\Lambda_0) + \dfrac{1}{2}B^N(\tau_i)'\Sigma^{r\pi}G_0\Sigma^{r\pi\prime}B^N(\tau_i) \\ \dot{B}^N(\tau_i) = \dfrac{1}{2}\tilde{B}^{N\prime}\tilde{\Sigma}^{r\pi}\tilde{G}\tilde{\Sigma}^{r\pi\prime}B^N(\tau_i) - (K' + \Lambda_1'\Sigma^{r\pi\prime})B^N(\tau_i) - [0\ 1\ 0]'\end{cases} \tag{4.15}$$

where we have defined the following matrices:

$$\tilde{B}^N = [B^N(\tau_i) \quad 0_{3\times 2}], \quad \tilde{\Sigma}^{r\pi} = [\Sigma^{r\pi} \quad \Sigma^{r\pi} \quad \Sigma^{r\pi}], \quad \tilde{G} = \begin{bmatrix} G_1 & 0_{5\times 5} & 0_{5\times 5} \\ 0_{5\times 5} & G_2 & 0_{5\times 5} \\ 0_{5\times 5} & 0_{5\times 5} & G_3 \end{bmatrix} \tag{4.16}$$

As the ODE's in (4.15) involve a matrix Riccati equation, no closed form solution is available (Polyanin and Zaitsev, 2002). Therefore we have to result to numerical estimations of the functions $A^N(\tau_i)$ and $B^N(\tau_i)$ in any numerical implementation of the CP2022 model. Lastly, we know that the payout of a bond that matures immediately should equal 1. Therefore we have $A^N(0) = 0$ and $B^N(0) = 0_{3\times 1}$.

In the original market, the agent can invest in a nominal stock, nominal money market account, and two nominal bonds. However, as the market contains five sources of risk, the original market is incomplete. Therefore, we add two assets to the market in which the agent cannot invest. In the dual market the trading constraints will be lifted so that we arrive at a complete market setup. The first asset we add to the original market is an inflation-indexed bond. Since the instantaneous real short rate R_t is affine in the state variables, we can utilize the results of Duffie and Kan (1996) according to the same logic we used to derive the price of a nominal bond. Consequently, we let the price of an inflation-linked bond expiring at time $t + \tau$ be equal to $P^R_{t+\tau}(t, X^s_t) = \mathbb{E}\left[\frac{\phi^R_{t+\tau}}{\phi^R_t}\middle|\mathcal{F}_t\right] = \exp\left(A^R(\tau) + B^R(\tau)'X^s_t\right)$. It should be noted that we postulate that the bond prices are a function of X^s_t only and not of X^o_t. At first, this result is nontrivial. The price of an inflation-linked bond is determined on the basis of the real pricing kernel. Since the real pricing kernel is influenced by the CPI, one would expect that the evolution of the inflation-indexed bond is also influenced by X^o_t. However, in the derivations it turns out that the elements of $B^R(\tau)$ corresponding to X^o_t are zero by construction (see (C.20)). Therefore it can be concluded that the inflation-indexed bond is a function of X^s_t only. Finally, it should be noted that we have defined the rest of the economy in nominal terms. Therefore, we also want to express the price of the inflation-indexed bond in nominal terms. The nominal price of the inflation-indexed is defined as follows $\hat{P}^R_{t+\tau}(t, X^s_t) = P^R_{t+\tau}(t, X^s_t)\Pi_t$. In Appendix C.1 we find the following dynamics of $\hat{P}^R_{t+\tau}(t, X^s_t)$:

$$\begin{aligned}\frac{d\hat{P}^R_{t+\tau}(t, X^s_t)}{\hat{P}^R_{t+\tau}(t, X^s_t)} &= \Big(r_t + (B^R(\tau)'\Sigma^{r\pi} + \sigma'_\Pi)(\Lambda_0 + \Lambda_1 X^s_t)\Big)dt \\ &\quad + \Big((B^R(\tau)'\Sigma^{r\pi} + \sigma'_\Pi)(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}\Big)\, dZ_t\end{aligned} \tag{4.17}$$

The functions $A^R(\tau) \in \mathbb{R}$ and $B^R(\tau) \in \mathbb{R}^{3\times1}$ are defined according to the following ODE's:

$$\begin{cases}\dot{A}^R(\tau) = B^R(\tau)'\Big(K\mathbb{E}X^s_\infty - \Sigma^{r\pi}(\Lambda_0 - G_0\sigma_\Pi)\Big) + \dfrac{1}{2}B^R(\tau)'\Sigma^{r\pi}G_0\Sigma^{r\pi\prime}B^R(\tau) \\ \qquad\qquad + \eta_\pi - \sigma'_\Pi\Lambda_0 \\ \dot{B}^R(\tau) = \dfrac{1}{2}\tilde{B}^{R\prime}\tilde{\Sigma}^{r\pi}\tilde{G}\tilde{\Sigma}^{r\pi}\tilde{\Sigma}^{r\pi\prime}B^R - (K' + \Lambda'_1\Sigma^{r\pi\prime} - \tilde{\sigma}'_\Pi\tilde{G}'\tilde{\Sigma}^{r\pi\prime})B^R(\tau) \\ \qquad\qquad - \Lambda'_1\sigma_\Pi - [0\ 1\ \ -1]'\end{cases} \tag{4.18}$$

where we have used the definitions of $\tilde{\Sigma}^{r\pi}$ and $\tilde{G}$ from (4.16) and we have defined the following matrices:

$$\tilde{B}^R = [B^R(\tau) \quad 0_{3\times2}], \quad \tilde{\sigma}_\Pi = \begin{bmatrix}\sigma_\Pi & 0_{5\times2} \\ \sigma_\Pi & 0_{5\times2} \\ \sigma_\Pi & 0_{5\times2}\end{bmatrix} \tag{4.19}$$

We conclude the asset mix by introducing an asset that trades the stochastic volatility. In the original market the agent cannot invest in this asset, whereas the trading constraint are lifted in the dual market. We introduce this asset in a way similar to the asset in Section 3.1 and thus we introduce $O_t = h(t, S_t, \nu_t)$. We know that the nominal price of this asset is defined as follows:

$$O_t = \mathbb{E}\left[\frac{\phi_T^N}{\phi_t^N} h(t, S_t, \nu_t) \Big| \mathcal{F}_t\right] \tag{4.20}$$

Using that $\phi_t^N O_t$ is a martingale, we find the following dynamics for O_t, again derivations can be found in Appendix C.1:

$$\begin{aligned} dO_t = &\Big(r_t O_t + (h_S S_t \sigma_S' + h_\nu \sigma_\nu')(\Lambda_0 + \Lambda_1 X_t^s)\Big) dt \\ &+ (h_S S_t \sigma_S' + h_\nu \sigma_\nu')(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}} dZ_t \end{aligned} \tag{4.21}$$

where $\sigma_\nu = [\omega\ 0\ 0\ 0\ 0]'$. Hence, σ_ν is the volatility of the stochastic volatility.

We conclude this section by combining all nominal asset in the vector $Y_t \in \mathbb{R}^{5x1}$, which supports further derivations:

$$Y_t = \begin{bmatrix} P_{t+\tau_1}^N(t, X_t^s) \\ P_{t+\tau_2}^N(t, X_t^s) \\ \hat{P}_{t+\tau}^R(t, X_t^s) \\ S_t \\ O_t \end{bmatrix} \tag{4.22}$$

The dynamics of Y_t are then specified as follows:

$$dY_t = \text{diag}(Y_t)\bigg(\Big(r_t + \Sigma_t(\Lambda_0 + \Lambda_1 X_t^s)\Big) dt + \Sigma_t(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}} dZ_t\bigg) \tag{4.23}$$

where $\Sigma_t \in \mathbb{R}^{5x5}$ represents the time dependent variance-covariance matrix of the complete financial market:

$$\begin{aligned} \Sigma_t &= \begin{bmatrix} B^N(\tau_1)'\Sigma^{r\pi} \\ B^N(\tau_2)'\Sigma^{r\pi} \\ B^R(\tau)'\Sigma^{r\pi} + \sigma_\Pi' \\ \sigma_S' \\ \frac{g_S S_t \sigma_S' + g_\nu \sigma_\nu}{O_t} \end{bmatrix} \\ &= \begin{bmatrix} (B^N(\tau_1)'\Sigma^{r\pi})_1 & (B^N(\tau_1)'\Sigma^{r\pi})_2 & (B^N(\tau_1)'\Sigma^{r\pi})_3 & 0 & 0 \\ (B^N(\tau_2)'\Sigma^{r\pi})_1 & (B^N(\tau_2)'\Sigma^{r\pi})_2 & (B^N(\tau_2)'\Sigma^{r\pi})_3 & 0 & 0 \\ (B^R(\tau)'\Sigma^{r\pi})_1 + \sigma_{\Pi,1} & (B^R(\tau)'\Sigma^{r\pi})_2 + \sigma_{\Pi,2} & (B^R(\tau)'\Sigma^{r\pi})_3 + \sigma_{\Pi,3} & \sigma_{\Pi,4} & \sigma_{\Pi,5} \\ \sigma_{S,1} & \sigma_{S,2} & \sigma_{S,3} & \sigma_{S,4} & \sigma_{S,5} \\ \left(\frac{h_S S_t \sigma_S' + h_\nu \sigma_\nu}{O_t}\right)_1 & \left(\frac{h_S S_t \sigma_S' + h_\nu \sigma_\nu}{O_t}\right)_2 & \left(\frac{h_S S_t \sigma_S' + h_\nu \sigma_\nu}{O_t}\right)_3 & \left(\frac{h_S S_t \sigma_S' + h_\nu \sigma_\nu}{O_t}\right)_4 & \left(\frac{h_S S_t \sigma_S' + h_\nu \sigma_\nu}{O_t}\right)_5 \end{bmatrix} \end{aligned} \tag{4.24}$$

Note that Σ_t is used for the variance-covariance matrix of the assets, whereas we defined Σ in (4.7) as the volatility matrix of the vector X_t.

4.2 Portfolio optimization problem

In this section, we present the portfolio optimization problem in the CP2022 model together with the reason why we cannot find an analytical optimal investment strategy. In the original incomplete market, the agent faces the following optimization problem:

$$\begin{aligned}&\sup_{\theta}\ \mathbb{E}\Big[\frac{(W_T/\Pi_T)^{1-\gamma}}{1-\gamma}\Big]\\&\text{s.t. } dW_t = W_t(r_t + \theta_t'\Sigma_t\Lambda_t)dt + W_t\theta_t'\Sigma_t dZ_t\end{aligned} \tag{4.25}$$

where $\theta_t \in \mathbb{R}^{5\times 1}$ represents an investment strategy in the original market. Note that the agent is interested in optimizing real terminal wealth, whereas investments can only be made in nominal assets. Due to the trading constraint on the inflation-linked bond and O_t, the optimization problem in (4.25) cannot be solved with the Martingale method. Therefore, we try to find the optimal strategy in the incomplete market via duality theory. Consequently, we want to solve the optimization problem in the dual market. In the dual market we have lifted the trading constraint on the inflation-linked bond and O_t. Consequently, we set the price of risk of the untraded Brownian motions equal to its price of risk in the original market plus the perturbation term a_t, in line with duality theory. Therefore, we postulate that the price of risk in the dual market looks as follows:

$$\begin{aligned}\tilde{\Lambda}_t &= ((\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}})^{-1}(\Lambda_0 + \Lambda_1 X_t^s + a_t)\\&= ((G_0 + \sum_i X_t^i G_i)^{\frac{1}{2}})^{-1}(\Lambda_0 + \bar{\Lambda}_1 X_t + a_t)\end{aligned} \tag{4.26}$$

where $a_t = [0\ 0\ a_{1,t}\ 0\ a_{2,t}]'$. The goal is then to set a_t such that there are no investments in the inflation-linked bond and O_t so that we have found the optimal strategy in the primal market. Hence, the perturbation term corresponds to the market price of risk of the unhedgeable risks. As a consequence of working with the perturbation term in the dual market, the dynamics of the nominal pricing kernel change accordingly:

$$\frac{d\tilde{\phi}_t^N}{\tilde{\phi}_t^N} = -r_t dt - \tilde{\Lambda}_t' dZ_t \tag{4.27}$$

with $\tilde{\phi}_0^N = 1$. Consequently, also the dynamics of the real pricing kernel change in the dual market. We define the real pricing kernel in the dual market as $\tilde{\phi}_t^R = \tilde{\phi}_t^N \Pi_t$. Hence, its dynamics can be found similarly to how we derived the dynamics of ϕ_t^R in Appendix C.1:

$$\begin{aligned}\frac{d\tilde{\phi}_t^R}{\tilde{\phi}_t^R} &= -(r_t - \pi_t - \eta_\pi + \sigma_\Pi{}'(\Lambda_0 + \Lambda_1 X_t^s + a_t))dt \\ &\quad - (\tilde{\Lambda}_t' - \sigma_\Pi{}'(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}})dZ_t \\ &= -\tilde{R}_t dt - \tilde{\Lambda}_t^{R\prime} dZ_t\end{aligned} \tag{4.28}$$

where $\tilde{\phi}_0^R = 1$. We have defined $\tilde{R}$ and $\tilde{\Lambda}_t^R$ as the real instantaneous short rate and real pricing kernel in the dual market, respectively. The modification of the real pricing kernel in the dual market is prompted by the adjusted definition of the nominal pricing kernel. However, the modification of the real instantaneous short rate is less obvious. If the optimization problem would be performed in nominal terms, the transition from the primal to the dual market would be entirely defined by the perturbation term. However, we define the optimization problem in real terms and consequently the real short rate also changes. Intuitively this can be explained by the fact that an agent cannot hedge inflation risk in the primal market. Hence, the instantaneous real short rate should be corrected for this in the dual market.

Now that we have defined all quantities that are related to the use of the perturbation term in the dual market, we can present the dynamic optimization problem the agent faces in the dual market if it optimizes real terminal dual wealth:

$$\begin{aligned}&\sup_{\theta}\ \mathbb{E}\Big[\frac{(\tilde{W}_T/\Pi_T)^{1-\gamma}}{1-\gamma}\Big] \\ &\text{s.t. } d\tilde{W}_t = \tilde{W}_t(r_t + \tilde{\theta}_t(a)'\Sigma_t\tilde{\Lambda}_t)dt + \tilde{W}_t\tilde{\theta}_t(a)'\Sigma_t dZ_t\end{aligned} \tag{4.29}$$

The budget constraint above specifies the evolution of dual wealth. It is thus dependent on price of risk particular to the dual market, defined in (4.26). We describe an investment strategy in the dual market with the vector $\theta_t(a) \in \mathbb{R}^{5\times 1}$. Each element of $\theta_t(a)$ corresponds to the position in the corresponding asset in Y_t. The remainder $1 - \sum_{i=1}^5 \theta_{i,t}(a_t)$ is invested in the money market account. We stress that the strategy in the dual market depends on the perturbation term a_t. If an optimal perturbation term could be found, the corresponding investment strategy would lead to the evolution of primal wealth. However, as will turn out later, we cannot solve the optimization problem in the dual market and thus we cannot find the optimal wealth process in the primal market. We can transfer the dynamic optimization problem to the following static problem:

$$\begin{aligned}&\sup_{W_T}\ \mathbb{E}\Big[\frac{(\tilde{W}_T/\Pi_T)^{1-\gamma}}{1-\gamma}\Big] \\ &\text{s.t. } \mathbb{E}\Big[\tilde{\phi}_T^N \tilde{W}_T\Big] = W_0\end{aligned} \tag{4.30}$$

If we define the Lagrangian $\mathcal{L}$ with corresponding Lagrange multiplier l, find the FOC and rewrite for $\tilde{W}^*_T$, we find the following expression for optimal terminal dual wealth:

$$\tilde{W}^*_T = \frac{W_0}{g_T}(\tilde{\phi}^N_T \Pi_T^{1-\gamma})^{-\frac{1}{\gamma}} \tag{4.31}$$

where we have defined g_T as $= \mathbb{E}\Big[(\tilde{\phi}^N_T \Pi_T)^{1-\frac{1}{\gamma}}\Big]$. The intermediate steps are the same as for the KNW model and thus we refer to Section 2.2 for further information on these steps. Since we treat wealth as a traded asset, we can find the optimal nominal dual wealth at time t by discounting the optimal terminal nominal wealth with the nominal pricing kernel in the dual market:

$$\tilde{W}^*_t = \frac{W_0}{g_T}(\tilde{\phi}^N_t)^{-\frac{1}{\gamma}}\Pi_t^{\frac{\gamma-1}{\gamma}}\,\mathbb{E}\left[(\frac{\tilde{\phi}^R_T}{\tilde{\phi}^R_t})^{\frac{\gamma-1}{\gamma}}|\mathcal{F}_t\right] \tag{4.32}$$

We thus need to find an explicit expression for $\tilde{P}(t,X_t) = \mathbb{E}\left[(\frac{\tilde{\phi}^R_T}{\tilde{\phi}^R_t})^{\frac{\gamma-1}{\gamma}}|\mathcal{F}_t\right]$ if we want to find the optimal wealth dynamics. In Appendix C.2 we find that we can rewrite the conditional expectation of interest to the following conditional expectation under the probability measure $\mathbb{Q}$, defined by the Radon-Nikodym derivative $\xi_t = \frac{dQ}{dP}_{|\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t \lambda_s'\lambda_s ds - \int_0^t \lambda_s' dZ_s\right\}$ with $\lambda_t = \frac{\gamma-1}{\gamma}\tilde{\Lambda}^R_t$:

$$\begin{aligned}\tilde{P}(t,X_t) &= \mathbb{E}\left[(\frac{\tilde{\phi}^R_T}{\tilde{\phi}^R_t})^{\frac{\gamma-1}{\gamma}}\middle|\mathcal{F}_t\right] = \mathbb{E}^Q\left[\frac{(\frac{\tilde{\phi}^R_T}{\tilde{\phi}^R_t})^{\frac{\gamma-1}{\gamma}}}{\xi_t}\middle|X_t\right] \\ &= \mathbb{E}^Q\left[\exp\left\{-\int_t^T\left(\frac{\gamma-1}{\gamma}\tilde{R}_s + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}^{R\prime}_s\tilde{\Lambda}^R_s\right)ds\right\}\middle|X_t\right]\end{aligned} \tag{4.33}$$

Consequently, if we apply the Feynman-Kac theorem to the conditional expectation under the probability measure $\mathbb{Q}$, we find the following PDE that $\tilde{P}(t,X_t)$ has to satisfy:

$$\begin{aligned}0 &= \dot{\tilde{P}} + \tilde{P}'_{X_t}(M + LX_t) + \frac{1}{2}\operatorname{tr}\left(\Sigma(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}\tilde{P}_{X_tX_t}(\Sigma(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}})'\right) \\ &\quad - \left(\frac{\gamma-1}{\gamma}\tilde{R}_t + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}^{R\prime}_t\tilde{\Lambda}^R_t\right)\tilde{P} - \frac{\gamma-1}{\gamma}\tilde{P}'_{X_t}\Sigma(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}\tilde{\Lambda}^R_t\end{aligned} \tag{4.34}$$

In the KNW and Heston model we found a similar PDE that describes the corresponding conditional expectation. We have solved the conditional expectation with the results of Duffie and Kan (1996) on affine yield models. We could utilize these results because the PDE's are affine (quadratic) functions of the state variables. Therefore, we could utilize the exponentially affine (quadratic) functional form for the conditional expectation, which is postulated by Duffie and Kan (1996). However, in the CP2022 model the conditional expectation involves the inner product of the real pricing kernel, which looks as follows:

$$\begin{aligned}\tilde{\Lambda}_t^{R\prime}\tilde{\Lambda}_t^R &= (\tilde{\Lambda}_t - (\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}\prime}\sigma_\Pi)'(\tilde{\Lambda}_t - (\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}\prime}\sigma_\Pi) \\ &= \tilde{\Lambda}_t'\tilde{\Lambda}_t - 2\sigma_\Pi'(\Lambda_0 + \bar{\Lambda}_1 X_t + a_t) + \sigma_\Pi'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\Pi \\ &= (\Lambda_0 + \bar{\Lambda}_1 X_t + a_t)'(\Gamma_0 + (X_t^s)_1\Gamma)^{-1}(\Lambda_0 + \bar{\Lambda}_1 X_t + a_t) \\ &\quad - 2\sigma_\Pi'(\Lambda_0 + \bar{\Lambda}_1 X_t + a_t) + \sigma_\Pi'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\Pi\end{aligned} \tag{4.35}$$

We thus see that the partial differential equation is dependent on the inverse of the stochastic variance matrix, i.e. $(\Gamma_0 + (X_t^s)_1\Gamma)^{-1}$, which is dependent on the stochastic volatility ν_t. Consequently, the PDE is not an affine (quadratic) function of X_t and thus we cannot apply the results of Duffie and Kan (1996) to find an analytical solution of the PDE[20]. We have studied several other methods that could possibly solve the conditional expectation, for example Laplace or Fourier transforms or the method of eigenfunction expansions. However, none of these methods can be applied to find an analytical expression for the conditional expectation. Furthermore, we have tried to find an analytical expression for the inverse of the stochastic variance matrix. However, due to the singularity of Γ_0 we cannot separate Γ_0 from $(X_t^s)_1\Gamma$ when calculating the inverse of $(\Gamma_0 + (X_t^s)_1\Gamma)^{-1}$. Consequently, we cannot find a method to work around the dependency of the inverse of the stochastic variance matrix on ν_t. Hence, although we did not provide a rigorous proof, we strongly suspect that it is not possible to analytically solve the conditional expectation that we are facing.

This belief is further supported by the findings in the literature on Heston's stochastic volatility model. In Section 3.2 we have assumed that the prices of risk are proportionally related to the square root of the stochastic variance. Without this assumption, it is not possible to analytically solve the portfolio optimization problem in Heston's stochastic volatility model (Nielsen and Jönsson, 2015). Within the framework of this thesis, this can be linked directly to the fact that the conditional expectation in the optimal wealth process cannot be solved analytically because it is no longer affine in ν_t. In other words, the derivations performed in Appendix B.2 to find an analytical expression for the conditional expectation, cannot be performed without the assumption that the prices of risk are proportional to $\sqrt{\nu_t}$. The nominal price of risk in the CP2022 model, given in (4.9), is clearly not proportional to the stochastic volatility matrix: it is inversely related to this matrix. As a consequence, also the real price of risk is not proportionally related to the stochastic volatility matrix. Hence, we argue that this structure of the price of risk causes the portfolio optimization problem to be unsolvable.

[20]Note that the term involving the trace operator in (4.34) also violates the affine structure of the PDE since it contains a cubical dependence on the stochastic volatility. We choose to demonstrate the breakdown of the PDE via the inner product of the pricing kernel because it directly links to the core of the problem, namely the dependence of the market prices of risk on the stochastic volatility matrix.

Since it is not possible to find the dynamics of the optimal wealth $\tilde{W}_t^*$, we cannot employ the fact that $\tilde{\phi}_t^N \tilde{W}_t^*$ is a martingale. Without these dynamics it is not possible to analytically find an optimal investment strategy in both the primal and dual market. It should be noted that we do have an analytical expression for the optimal dual wealth. However, as we cannot specify a dual optimal investment strategy, we cannot link the dual wealth to primal wealth.

We want to end this section with a note on the HJB method. In the framework of this thesis we have used the Martingale method in combination with duality theory to solve the optimal investment problem in the incomplete market. The HJB approach works in incomplete markets, and thus no artificial completion of the original market is needed. Consequently, the HJB approach provides an alternative framework to find the optimal investment strategy in the original incomplete market. In Heston's stochastic volatility model the portfolio optimization problem can also not be solved with the HJB method without the proportionality assumption on the prices of risk (Nielsen and Jönsson, 2015). Therefore, we expect that the portfolio optimization problem in the model of Commissie Parameters (2022) is also unsolvable if the HJB method is used. However, further research is needed to formalize this.

4.3 Possible alternative investment strategies

In the previous section we argued that no analytical optimal investment strategy can be found in the model of Commissie Parameters (2022). The goal of this section is to present an estimate for the optimal strategy. We will do this by adjusting the PDE that we could not solve in Section 4.2. By means of an appropriate modification we can apply the results of Duffie and Kan (1996) on affine yield models. In Section 4.2, we define this PDE in the context of the dual market. If we apply the modification of the PDE in the dual market to find an estimated dual strategy it will remain unclear what the corresponding true optimal dual strategy is. As a consequence it will not be possible to determine the optimal perturbation term and thus the estimated strategy in the dual market cannot be linked to a strategy in the primal market. Therefore, we forget about the duality framework in this section. From now on we assume that we are in a complete market setup. In other words, we take the market as given in Section 4.1, without trading constraints. Hence, we face a market in which no trading constraints on the inflation-linked bond and O_t exist so that the original market is complete. In this complete market setup we know that the true the optimal wealth at time t looks as follows:

$$W_t^* = \frac{W_0}{g_T} (\phi_t^N)^{-\frac{1}{\gamma}} \Pi_t^{\frac{\gamma-1}{\gamma}} \mathbb{E}\left[(\frac{\phi_T^R}{\phi_t^R})^{\frac{\gamma-1}{\gamma}} | \mathcal{F}_t \right] \tag{4.36}$$

where the dynamics of Π_t, ϕ_t^N, and ϕ_t^R are given in (4.6), (4.12), and (4.13), respectively. Furthermore, we define g_T as $= \mathbb{E}\Big[(\phi_T^N \Pi_T)^{1-\frac{1}{\gamma}}\Big]$. We stress that we do not use the pricing kernels specific to the dual market in this section.

The primary difficulty in solving the portfolio optimization problem in the complete market is to find an explicit expression for the conditional expectation in (4.36). We define the true conditional expectation as $P(t, X_t) = \mathbb{E}\left[(\frac{\phi_T^R}{\phi_t^R})^{\frac{\gamma-1}{\gamma}}|\mathcal{F}_t\right]$. Following the same line of argument as we did in Section 4.2 for the dual market, we find the following PDE that describes the true value of $P(t, X_t)$ in the new complete market setup:

$$
\begin{aligned}
0 = \dot{P} + P'_{X_t}(M + LX_t) + \frac{1}{2}\operatorname{tr}\left(\Sigma(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}} P_{X_tX_t}(\Sigma(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}})'\right) \\
- \left(\frac{\gamma-1}{\gamma}R_t + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\Lambda_t^{R\prime}\Lambda_t^R\right)P - \frac{\gamma-1}{\gamma}P'_{X_t}\Sigma(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}\Lambda_t^R
\end{aligned}
\tag{4.37}
$$

The results of Duffie and Kan (1996) cannot be applied directly to the PDE above for two reasons: (i) the inner product of the real pricing kernel is dependent on the inverse of the stochastic volatility matrix and (ii) the term involving the trace operator contains a cubical relation with the stochastic volatility. The affine quadratic structure of the PDE is violated by these two quantities and thus the PDE needs to be modified so that it becomes an affine quadratic function in X_t. The core of the problems that arise when solving the portfolio optimization problem in the model of Commissie Parameters (2022) is the structure of the price of risk. The most elegant solution to make the PDE affine quadratic would therefore be to modify the price of risk uniformly across the whole PDE. We stress that this implies the use of a different price of risk only in the PDE. Hence, we do not alter the rest of the model because there only are difficulties when solving the PDE. With the right modification, the inner product of the real pricing kernel can become affine quadratic in X_t. However, such a modification also influences the last part of the PDE ($\frac{\gamma-1}{\gamma}P'_{X_t}\Sigma(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}\Lambda_t^R$). This term measures the correlation between the state variables and the real pricing kernel. It turns out that no modification of the price of risk exists that simultaneously forces the inner product of Λ_t^R to become affine quadratic in X_t whilst maintaining the affine quadratic structure of the correlation term. Once a structure is assumed that fixes the structure of $\Lambda_t^{R\prime}\Lambda_t^R$, a problem with the correlation term arises. Furthermore, working with a modified price of risk will not directly solve the cubical dependence of the trace operator on ν_t. This cubical dependence is caused by the structure of the dynamics of X_t. However, the structure of these dynamics is indirectly caused by the structure of the price of risk. Hence, assuming an alternative form for the pricing kernel will not directly remove the cubital dependence of the trace operator on ν_t.

We thus see that a uniform modification of the price of risk does not make the PDE affine quadratic in X_t. Therefore, we propose a different modification of the PDE. First of all, we do not change the correlation term since it respects the affine quadratic structure of the PDE. We thus want to find the smallest adjustment of the model that makes the inner product of the real pricing kernel and the trace operator affine quadratic in X_t. Therefore we propose a modification of the stochastic volatility matrix that makes it constant. This constant matrix will replace the stochastic volatility matrix in both the inner product and the trace operator. We define $\tilde{G}^\infty$ below:

$$\tilde{G}^\infty = (\Gamma_0 + \mathbb{E}\nu_\infty \Gamma) \tag{4.38}$$

We thus see that $\tilde{G}^\infty$ serves as the constant approximation of the stochastic volatility matrix, where we partly maintain the dependence on ν_t by assuming the matrix depends on the long-run mean of the stochastic variance process. Consequently, we define estimation of the real price of risk that is driven by $\tilde{G}^\infty$ as $\tilde{\Lambda}_t^R$:

$$\tilde{\Lambda}_t^R = ((\tilde{G}^\infty)^{\frac{1}{2}})^{-1}(\Lambda_0 + \bar{\Lambda}_1 X_t) - (\tilde{G}^\infty)^{\frac{1}{2}\prime}\sigma_\Pi \tag{4.39}$$

Note that in Section 4.2 we defined $\tilde{\Lambda}_t^R$ as the real price of risk in the dual market. In the complete market setup of this section we define $\tilde{\Lambda}_t^R$ as the real pricing kernel specified by the constant volatility matrix rather than the stochastic volatility matrix. On the basis of (4.39) we propose to replace the inner product $\Lambda_t^{R\prime}\Lambda_t^R$ in the PDE with $\tilde{\Lambda}_t^{R\prime}\tilde{\Lambda}_t^R$. This modification removes the dependence on the inverse of ν_t. Consequently, this part of the PDE exhibits an affine quadratic structure in X_t.

Now that we have modified the first term that makes the PDE unsolvable, we want to modify the trace term so that also here an affine quadratic structure in X_t can be assumed. Before doing so, we define $\tilde{P}(t, X_t)$ as the functional form that solves the modified PDE. If we assume that $\tilde{P}(t, X_t)$ is affine quadratic in X_t, we can assume $\tilde{P}(t, X_t) = \exp\left(\tilde{A}(t) + \tilde{B}(t)'X_t + X_t'\tilde{C}(t)X_t\right)$ by the results of Duffie and Kan (1996). Here, $\tilde{A}(t) \in \mathbb{R}$, $\tilde{B}(t) \in \mathbb{R}^{5\times 1}$ and $\tilde{C}(t) \in \mathbb{R}^{5\times 5}$. We know that the derivatives of $\tilde{P}(t, X_t)$ are given as follows:

$$\begin{cases} \dot{\tilde{P}} = \tilde{P}(\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)'X_t + X_t'\dot{\tilde{C}}(t)X_t) \\ \tilde{P}_X = \tilde{P}(\tilde{B}(t) + 2\tilde{C}(t)X_t) \\ \tilde{P}_{XX} = \tilde{P}((\tilde{B}(t) + 2\tilde{C}(t)X_t)(\tilde{B}(t) + 2\tilde{C}(t)X_t)' + 2\tilde{C}(t)) \end{cases} \tag{4.40}$$

If we replace all entries in the trace operator where the the stochastic volatility matrix comes back with the constant volatility matrix and impose the affine quadratic structure on the PDE, we can rewrite the trace term as follows:

$$\begin{aligned} &\operatorname{tr}\left(\Sigma(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}\tilde{P}_{X_tX_t}(\Sigma(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}})'\right) = \\ &\operatorname{tr}\left(\tilde{B}'\Sigma\tilde{G}^\infty\Sigma'\tilde{B} + 4\tilde{B}'\Sigma\tilde{G}^\infty\Sigma'\tilde{C}X_t + 4X_t'\tilde{C}'\Sigma\tilde{G}^\infty\Sigma'\tilde{C}X_t + 2\Sigma(\tilde{G}^\infty)^{\frac{1}{2}}\tilde{C}(\Sigma(\tilde{G}^\infty)^{\frac{1}{2}})'\right) \end{aligned} \tag{4.41}$$

We thus see that the trace operator becomes affine quadratic in X_t when using the constant volatility matrix.

Because of the explained modifications, the results of Duffie and Kan (1996) can be applied to the modified PDE. We therefore postulate the following estimated functional form of the conditional expectation:

$$\tilde{P}(t, X_t) = \mathbb{E}\left[\left(\frac{\phi_T^R}{\phi_t^R}\right)^{\frac{\gamma-1}{\gamma}} \middle| \mathcal{F}_t \right] \approx \exp\left(\tilde{A}(t) + \tilde{B}(t)' X_t + X_t' \tilde{C}(t) X_t \right) \tag{4.42}$$

In (C.33) and (C.34) we define the system of ODE's that describes $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$. We know that the terminal condition of our true PDE should equal 1, i.e. $\mathbb{E}\left[\left(\frac{\phi_T^R}{\phi_T^R}\right)^{\frac{\gamma-1}{\gamma}} | \mathcal{F}_t \right] = 1$. Hence, we assume the terminal conditions of $\tilde{P}(t, X_t)$ to respect this conditions. We thus assume that $\tilde{A}(T) = 0$, $\tilde{B}(T) = 0_{5\times 1}$ and $\tilde{C}(T) = 0_{5\times 5}$. Note that no closed-form solution is available for the system since it involves a matrix Riccati equation.

With the estimated expression for the conditional expectation we have an (estimated) SDE for each element of the optimal wealth formula in (4.36). Therefore, we can estimate the optimal strategy in the same way as in the dual market for the KNW model, in Section 2.2. Using the estimated value of the conditional expectation in the wealth formula in (4.36) allows us to find estimated dynamics of $W_t^* \phi_t^N$. We know that the corresponding true dynamics should be a martingale. However, we rely on an estimation of the conditional expectation and thus the estimated dynamics of $W_t^* \phi_t^N$ will not be a martingale. Therefore, we modify the dynamics of $W_t^* \phi_t^N$ in the same way as we modified the underlying PDE. In other words, we change the dynamics of $W_t^* \phi_t^N$ on the places where the inner product and the trace operator come back so that the dynamics become a martingale again. Without this slight abuse of Itô calculus, $W_t^* \phi_t^N$ is not a martingale and thus no optimal strategy can be found.

The budget constraint provides an alternative definition of the optimal wealth so that we can find a second way to describe the dynamics of $W_t^* \phi_t^N$. The budget constraint does not have its origins in the conditional expectation we estimate in this section and thus Itô calculus can be applied here directly. By equating the volatility terms of the expressions of discounted optimal wealth ultimately we find the following estimated optimal strategy:

$$\begin{aligned} \tilde{\theta}_t^* = &\frac{1}{\gamma} \left(\Sigma_t^{-1}\right)^\top \left(\Gamma_0 + (X_t^s)_1 \Gamma\right)^{-1} \left(\Lambda_0 + \Lambda_1 X_t^s\right) + \frac{\gamma - 1}{\gamma} \left(\Sigma_t^{-1}\right)^\top \sigma_\Pi \\ &+ \left(\Sigma_t^{-1}\right)^\top \Sigma^\top \left(\tilde{B}(t) + 2\tilde{C}(t) X_t\right) \end{aligned} \tag{4.43}$$

We refer to Appendix C.3 for further details on the derivations. Note that the strategy is defined with a tilde to stress that it is an estimation of the true optimal strategy. We see that the optimal strategy resembles the optimal strategy in the KNW market since it contains mean-variance optimal demand and two hedge demands that are equal to the hedge demands in the KNW model. In contrast to the strategy in the model of Koijen et al. (2010), the mean-variance optimal demand is corrected with the inverse of the stochastic variance matrix, which can be viewed as a correction of the strategy for the stochastic volatility. Note that an extra hedge demand caused by the stochastic volatility would be logical. This term is possibly not captured by the estimation. It is reasonable to assume that at least the inverse of the variance-covariance matrix of the assets will enter the true optimal strategy.

The estimated strategy in (4.43) can be used in the financial market we defined in Section 4.1. This leads to estimated values of the optimal terminal wealth. The expression for optimal wealth in (4.36) gives a method to directly simulate the true optimal terminal wealth, without using the estimated PDE. Consequently, we can compare the welfare generated with the estimated strategy with the welfare generated with the optimal strategy. However, in the numerical implementation of the estimated strategy we encounter two numerical problems. The biggest problem is related to the variance-covariance matrix of the assets, defined in (4.24). Note that this matrix depends on a specification of the asset we added to the market, O_t. In the simulations we have assumed $O_t = 2\nu_t$ for tractability. In general any asset that respects the dependence on ν_t can be chosen. All specifications we have examined do not solve the problem we will describe. For the parameter estimates in the scenario sets of the first and second quarter of 2025 we find that the time-dependent variance-covariance matrix is nearly singular for each time point. Hence, its inverse converges to infinity, causing the estimated strategy to converge to infinity. When controlling for this singularity, by for example dividing the inverse of the matrix by a factor 1000, the wealth does not blow up in most of the simulated scenarios. Any other problems are caused by the dependence of the strategy on the inverse of the stochastic volatility matrix. In some scenarios the simulated volatility converges to zero. We know that $(\Gamma_0 + (X_t^s)_1\Gamma)_{1,1}^{-1}$ equals ν_t. Because the volatility matrix is diagonal, its inverse can be found by taking the inverse of the diagonal elements. In the scenarios where ν_t converges to zero, the corresponding element of the inverse of the stochastic volatility matrix will thus converge to infinity. This causes the estimated strategy to blow up to infinity.

We thus find an estimated strategy that exhibits a relatively logical structure, based on the findings in Chapter 2. However, when numerically implementing the strategy problems arise. The numerical problems are inherent to the model setup. The inverse of the stochastic volatility matrix will converge to zero in some scenarios in any numerical implementation. Hence, explosion of its inverse can always occur. Although the variance-covariance matrix of the assets depends on O_t and the bond maturities, we find the matrix to be singular for each possible set of bond durations and choice of O_t. Therefore, it seems that also the singularity of this matrix can be seen as a numerical artefact of the model. The difficulty with the numerical problems is that we do not know the true optimal strategy and thus we do not know whether the terms causing the problems come back in the true optimal KNW strategy in this form. Hence, it remains unanswered whether the estimated strategy blows up because of a wrong estimation or because of artifacts of the model.

In summary of this section we conclude that we could not perform a numerical welfare comparison between the true optimal and estimated terminal wealth. However, we argue that the strategy in (4.43) and the underlying modifications made to be model can be studied further, which can possibly lead to a strategy that can be implemented numerically. Furthermore, other simulation methods can be studied so that the numerical problems becomes less persistent. For example, refinements of the Euler schemes can be used. Therefore, we chose to present the results on the estimated strategy although we did not calculate the corresponding welfare. In this way, it can serve as the basis for future research.

Chapter 5

Conclusion and future research

5.1 Conclusion

We introduced the financial market models of Koijen et al. (2010), Heston (1993), and Commissie Parameters (2022) with the primary objective of analyzing the fundamental welfare risks inherent in the risk preference research. The first crucial question underlying the risk preference research concerns the welfare effects of using Merton's mean-variance optimal strategy when the economy differs from the Black-Scholes financial market. The second risk is related to the use of CRRA preferences. If the risk preference parameter of an agent is wrongly measured, a suboptimal strategy will be implemented, leading to welfare losses. To investigate these welfare effects, we started with finding the optimal investment strategy in the incomplete markets with the Martingale method. To facilitate the use of the Martingale method, we introduced duality theory, which is introduced in the literature by Karatzas et al. (1991), Cvitanic and Karatzas (1992), Xu and Shreve (1992), and Kamma (2023). The duality theory allowed the optimal strategy in the model of Koijen et al. (2010) and Heston (1993) to be found without ex-post modifications of the financial markets. We argue that the combination of the Martingale method and duality theory proposes a methodological refinement of the existing literature on portfolio optimization problems in incomplete markets.

Prompted by duality theory, we artificially completed the financial market of Koijen et al. (2010) with an inflation-linked bond in which the agent is not allowed to invest. Via the dual optimization problem we found the optimal investment strategy in the incomplete market. Although optimal investment in related models is studied, the optimal strategy in the model of Koijen et al. (2010) was not publicly available, and thus we contribute to literature by defining the optimal investment strategy in the incomplete KNW market. The optimal strategy is highly sensitive with respect to the model parameters, which is in line with the literature on affine yield models, for example by Balter et al. (2021). Independent of the parameters we found extreme allocations to the two nominal bonds. To investigate the welfare effects of using Merton's mean-variance optimal strategy, we defined two strategies that resemble the mean-variance optimal strategy. The myopic strategy does not take hedge demands into account but accounts for time variation in prices of risk, whereas the static strategy does not account for the time variation in market prices of risk. The static strategy can thus be seen as a constant myopic strategy. For both strategies, we find welfare losses statistically significant different from zero for each value of the risk aversion parameter, independent of the model parameters. Depending on the suboptimal strategy and parameters, a different relation with the risk aversion parameter is found. In the worst cases welfare losses up to 90% can be found. In general, the myopic strategy leads to smaller welfare losses than the static strategy. Furthermore, welfare losses from the myopic strategy tend to increase with γ, whereas welfare losses from the static strategy tend to decrease with γ. With respect to using an incorrect risk aversion parameter, we find that welfare losses increase if the true value lies further away from the measured value. For true and measured risk aversion parameters that are sufficiently close, small welfare losses are found, possibly not statistically significant different from zero. In general, underestimations of the true risk preference parameter will lead to higher welfare losses than overestimations. An overestimation of the true risk preference parameter of at most two will in general not lead to welfare losses as high as the welfare losses from myopic/static investment, if the true γ is larger than two. On the other hand, underestimations of size two can lead to large welfare losses.

We artificially completed the financial market model of Heston (1993) by adding an additional asset in which the agent is not allowed to invest. This allowed the optimal investment strategy in the incomplete financial market to be found via the dual optimization problem. Although the optimization problem in Heston's model has been studied in the literature, we argue that the duality theory proposes a methodological refinement of the methods used in literature. For example, most papers do not reduce the demand of an additional asset to zero. Consequently, these papers present no optimal strategy in the incomplete market. On the other hand, papers that use a dynamic programming approach define the optimal strategy in the incomplete market. However, we argue that the Martingale method in combination with duality theory is technically less demanding than a dynamic programming method. We found the optimal allocation to the stock to be non-stochastic but time-dependent. In general, a small difference between the optimal strategy and the mean-variance optimal strategy is concluded. Hence, a myopic investor in the model of Heston (1993) faces significantly smaller welfare losses than a myopic investor in the model of Koijen et al. (2010). An investor that uses an incorrect risk aversion parameter typically experiences large welfare losses and a similar pattern as in the KNW model, under the same misspecification of γ. One notable distinction between the welfare losses arising from using an incorrect risk aversion parameter in the models of Heston (1993) and Koijen et al. (2010) is that, in Heston's model, a greater fraction of the losses are not statistically distinguishable from zero. This especially happens if the true risk aversion parameter is large and the measured risk aversion parameter is close to the true value. It can be explained by the fact that the optimal strategies in Heston's model converge to each other more rapidly if γ increases than the optimal strategies in the model of Koijen et al. (2010). This can be explained by the fact that the mean-variance optimal demand is constant in combination with the magnitude of the hedge demand being small in the model of Heston (1993). Since the mean-variance demand is inversely related to γ, strategies will rapidly converge to each other for high values of the risk aversion parameter, which is not the case in the KNW model. Hence, small misspecifications of γ typically lead to lower welfare losses in the Heston model than in the KNW model.

Finally, we studied the portfolio optimization problem of a CRRA investor that receives utility from terminal wealth in the model of Commissie Parameters (2022). No findings in literature are known on optimal investment in the model of Commissie Parameters (2022). We argue that, due to the inverse dependence of the prices of risk on the stochastic volatility, no optimal investment strategy can be found analytically on the basis of currently known techniques. This argument is based on the findings in the model of Heston (1993), in which also no optimal strategy could be found if the prices of risk would have an inverse dependence on the stochastic volatility. Although we do not expect the portfolio optimization problem to solvable in the model of Commissie Parameters (2022), we did not provide a rigorous proof on why the portfolio optimization problem breaks down. Hence, further research could possibly lead to different results. We concluded the findings on the CP2022 model with an analytical estimation of the optimal strategy. Although we have found an estimated optimal strategy, numerical complications arise when implementing the strategy and thus no welfare comparison between the optimal wealth and the wealth generated by the estimated strategy is performed. We chose to present the estimated strategy so that the results can serve as a basis for future research.

5.2 Future research

A key direction for future research emerging from this thesis is the optimal investment problem within the CP2022 model. First of all, further research can be conducted on the optimal analytical strategy. Hypothetically, new techniques that work around the problems we faced in this thesis could be invented. If not, a mathematical rigorous proof on why the problem cannot be solved could possibly be found. We suspect that the portfolio optimization problem also cannot be solved when a dynamic programming method is used, which can be investigated in future research. Also outside the context of the optimal analytical strategy, the CP2022 model could be analyzed further. We presented an analytical estimation of the optimal strategy, which could not be implemented due to numerical problems. In the first place, the numerical problems with the current strategy can possibly be solved. For example by a refinement of the implementation of the continuous processes. Secondly, alternative estimated strategies can be postulated. If a numerical implementation then turns out to be possible, the generated welfare can be compared with the true optimal welfare. Outside the context of analytically estimating the optimal strategy, an algorithm that numerically finds the optimal strategy could be created. Lastly, one could adjust the CP2022 model and consequently set up a new model in which stochastic interest rates, inflation, and volatility are taken into account. If an alternative but related model could be found such that the portfolio problem in this new model can be solved analytically, one could perform welfare analysis in this alternative model.

Finally, we focused on an agent that receives utility from terminal wealth alone. A more comprehensive version of the problem can be studied. For example, consumption can be added to the problem. Furthermore, reference levels could be imposed or stochastic processes for labor income can be assumed, similar to the research of Bodie et al. (1992) or Cocco et al. (2005). In this thesis we assume the use of CRRA preferences. The effects of different utility functions in the models of Koijen et al. (2010) and Heston (1993) could also provide relevant insights. For example, Yang and Pelsser (2023) compare the investment strategy in Heston's model based on CRRA preferences and SAHARA preferences.

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Appendix A

Stochastic interest rate

A.1 Financial market

We define the real pricing kernel as $\phi_t^R = \phi_t^N \Pi_t$ and $d[X,Y]_t$ as the quadratic covariation between X and Y, i.e. $d[X,Y]_t = \lim_{\Delta t \to 0} \sum_{j=0}^{t} (X_{t_{j+1}} - X_j)(Y_{t_{j+1}} - Y_j)$. An application of Itô's Lemma yields:

$$\begin{aligned} \frac{d\phi_t^R}{\phi_t^R} &= \frac{\phi_t^N d\Pi_t}{\phi_t^N \Pi_t} + \frac{\Pi_t d\phi_t^N}{\phi_t^N \Pi_t} + \frac{d[\phi^N, \Pi]_t}{\phi_t^N \Pi_t} \\ &= \frac{d\Pi_t}{\Pi_t} + \frac{d\phi_t^N}{\phi_t^N} - \sigma_\Pi' \Lambda_t dt \\ &= -r_t dt + \pi_t dt - \sigma_\Pi' \Lambda_t dt - \Lambda_t' dZ_t + \sigma_\Pi' dZ_t \\ &= -(r_t - \pi_t + \sigma_\Pi' \Lambda_t) dt - (\Lambda_t' - \sigma_\Pi') dZ_t \end{aligned} \tag{A.1}$$

To derive the price dynamics of a nominal bond maturing at time τ_i, we denote the price by $P = P_{t+\tau_i}^N(t, X_t) = \mathbb{E}\left[\frac{\phi_{t+\tau_i}^N}{\phi_t^N} \middle| \mathcal{F}_t\right] = \exp\left(A^N(\tau_i) + B^N(\tau_i)' X_t\right)$. To support further derivations, we first calculate the derivatives of P. Note that the time to maturity τ_i decreases as t increases so that the time derivative of P takes a negative sign when expressed as function of τ_i.

$$\begin{cases} \dot{P} = P(-\dot{A}^N(\tau_i) - \dot{B}^N(\tau_i)' X_t) \\ P_X = P B^N(\tau_i) \\ P_{XX} = P B^N(\tau_i) B^N(\tau_i)' \end{cases} \tag{A.2}$$

Then we can use the derivatives in (A.2) to derive the dynamics of $P^N_{t+\tau_i}(t, X_t)$:

$$
\begin{aligned}
dP^N_{t+\tau_i}(t, X_t) &= d\exp\Big(A^N(\tau_i) + B^N(\tau_i)'X_t\Big) \\
&= \dot{P}dt + P'_X dX_t + \frac{1}{2}P_{XX}d[X, X]_t + \frac{1}{2}\ddot{P}d[t, t]_t + P_{Xt}d[X, t]_t \\
&= \dot{P}dt + P'_X\left(-K_X X_t dt + \Sigma_X dZ_t\right) + \frac{1}{2}\Sigma_X P_{XX}\Sigma'_X dt \\
&= (\dot{P} - P'_X K_X X_t + \frac{1}{2}tr(\Sigma_X P_{XX}\Sigma'_X))dt + P'_X\Sigma_X dZ_t \\
&= P^N_{t+\tau_i}(t, X_t)\{(-\dot{A}^N(\tau_i) - \dot{B}^N(\tau_i)'X_t - B^N(\tau_i)'K_X X_t \\
&\quad + \frac{1}{2}B^N(\tau_i)'\Sigma_X\Sigma'_X B^N(\tau_i))dt + B^N(\tau_i)'\Sigma_X dZ_t\}
\end{aligned}
\tag{A.3}
$$

where we have used that $\mathrm{tr}(\Sigma_X B^N(\tau_i)B^N(\tau_i)'\Sigma'_X) = B^N(\tau_i)'\Sigma_X\Sigma'_X B^N(\tau_i)$ by the properties of the trace operator and the fact that $B^N(\tau_i)'\Sigma_X\Sigma'_X B^N(\tau_i) \in \mathbb{R}$. Then, by FTAP we know that the product of the pricing kernel and the bond price must be a martingale (Schumacher, 2020). Therefore, we determine the dynamics of $\phi^N_t P^N_{t+\tau_i}(t, X_t)$:

$$
\begin{aligned}
d\phi^N_t P^N_{t+\tau_i}(t, X_t) &= \phi^N_t dP^N_{t+\tau_i}(t, X_t) + P^N_{t+\tau_i}(t, X_t)d\phi^N_t + d[\phi^N_t, P^N_{t+\tau_i}(t, X_t)]_t \\
&= \phi^N_t P\{(-\dot{A}^N(\tau_i) - \dot{B}^N(\tau_i)'X_t - B^N(\tau_i)'K_X X_t \\
&\quad + \frac{1}{2}B^N(\tau_i)'\Sigma_X\Sigma'_X B^N(\tau_i))dt + B^N(\tau_i)'\Sigma_X dZ_t\} \\
&\quad + \phi^N_t P\left(-r_t dt - \Lambda'_t dZ_t\right) - \phi^N_t P B^N(\tau_i)'\Sigma_X\Lambda_t dt \\
&= \phi^N_t P^N_{t+\tau_i}(t, X_t)\{(-\dot{A}^N(\tau_i) - \dot{B}^N(\tau_i)'X_t - B^N(\tau_i)'K_X X_t \\
&\quad + \frac{1}{2}B^N(\tau_i)'\Sigma_X\Sigma'_X B^N(\tau_i) - r_t - B^N(\tau_i)'\Sigma_X\Lambda_t)dt \\
&\quad + (B^N(\tau_i)'\Sigma_X - \Lambda'_t)dZ_t\}
\end{aligned}
\tag{A.4}
$$

By applying the Martingale property of the process in (A.4), we find the following PDE that bond prices must satisfy:

$$
\begin{aligned}
0 &= (-\dot{A}^N(\tau_i) - \dot{B}^N(\tau_i)'X_t) - B^N(\tau_i)'K_X X_t + \frac{1}{2}B^N(\tau_i)'\Sigma_X\Sigma'_X B^N(\tau_i) \\
&\quad - (\delta_{0,r} + \delta'_{1,r}X_t) - B^N(\tau_i)'\Sigma_X(\Lambda_0 + \Lambda_1 X_t)
\end{aligned}
\tag{A.5}
$$

We can split the PDE in (A.5) in two parts: a stochastic part depending on X_t and a non-stochastic part that is independent of X_t (Draper, 2012). As both parts should equate to zero, we find the following two ODE's that fully describe the bond price:

$$
\begin{cases}
\dot{A}^N(\tau_i) = -\delta_{0,r} + \frac{1}{2}B^N(\tau_i)'\Sigma_X\Sigma'_X B^N(\tau_i) - B^N(\tau_i)'\Sigma_X\Lambda_0 \\
\dot{B}^N(\tau_i) = -\delta_{1,r} - (K'_X + \Lambda'_1\Sigma'_X)B^N(\tau_i)
\end{cases}
\tag{A.6}
$$

Koijen et al. (2010), Draper (2012) and Kamma and Pelsser (2022) show, among others, that the ODE's in (A.6) obey the following solution:

$$\begin{cases} A^N(\tau_i) = \int_0^{\tau_i} \dot{A}^N(s)ds \\ B^N(\tau_i) = (K_X' + \Lambda_1'\Sigma_X')^{-1}\left\{\exp(-(K_X' + \Lambda_1'\Sigma_X')\tau_i) - \mathrm{I}_{2x2}\right\}\delta_{1,r} \end{cases} \tag{A.7}$$

Finally, combining the SDE of $P^N_{t+\tau_i}(t, X_t)$ from (A.3) with the definitions of $\dot{A}^N(\tau_i)$ and $\dot{B}^N(\tau_i)$ from (A.6) lead to the full specification of the nominal bond price dynamics:

$$\begin{aligned} \frac{dP^N_{t+\tau_i}(t, X_t)}{P^N_{t+\tau_i}(t, X_t)} &= \{-\dot{A}^N(\tau_i) - \dot{B}^N(\tau_i)'X_t - B^N(\tau_i)'K_X X_t \\ &\quad + \frac{1}{2}B^N(\tau_i)'\Sigma_X\Sigma_X'B^N(\tau_i)\}dt + B^N(\tau_i)'\Sigma_X dZ_t \\ &= \{\delta_{0,r} - \frac{1}{2}B^N(\tau_i)'\Sigma_X\Sigma_X'B^N(\tau_i) + B^N(\tau_i)'\Sigma_X\Lambda_0 \\ &\quad - (-\delta_{1,r} - (K_X' + \Lambda_1'\Sigma_X')B^N(\tau_i))'X_t - B^N(\tau_i)'K_X X_t \\ &\quad + \frac{1}{2}B^N(\tau_i)'\Sigma_X\Sigma_X'B^N(\tau_i)\}dt + B^N(\tau_i)'\Sigma_X dZ_t \\ &= \left(r_t + B^N(\tau_i)'\Sigma_X\Lambda_t\right)dt + B^N(\tau_i)'\Sigma_X dZ_t \end{aligned} \tag{A.8}$$

As $X_0 = \begin{bmatrix}0 \; 0\end{bmatrix}'$, we have $P^N_{\tau_i}(0, X_0) = \exp\left(A^N(\tau_i) + B^N(\tau_i)'X_0\right) = \exp(A^N(\tau_i))$.

To derive the dynamics of the inflation-linked bond maturing at time τ we follow a similar procedure as for the nominal bonds. By the Duffie and Kan (1996) results on affine yield models we know that the functional form of the bond price is given by $P^R_{t+\tau}(t, X_t) = \mathbb{E}\left[\frac{\phi^R_{t+\tau}}{\phi^R_t}\middle|\mathcal{F}_t\right] = \exp\left(A^R(\tau_i) + B^R(\tau_i)'X_t\right)$. Hence, the dynamics of $P^R_{t+\tau}(t, X_t)$ follow a similar structure as the dynamics of $P^N_{t+\tau_i}(t, X_t)$ in (A.3). Note however that this expresses the price of the inflation-linked bond in real terms. As the rest of the economy is defined in nominal terms, we also would like to express the price of the inflation-linked bond in nominal terms. Therefore we are interested in $\hat{P}^R_{t+\tau}(t, X_t) = P^R_{t+\tau}(t, X_t)\Pi_t$. To arrive at the dynamics of $\hat{P}^R_{t+\tau}(t, X_t)$ we first find the dynamics of $P^R_{t+\tau}(t, X_t)$. To this end, we start with deriving the dynamics of $\phi^R_t P^R_{t+\tau}(t, X_t)$. If we now let $P = P^R_{t+\tau}(t, X_t)$ we find the following dynamics:

$$\begin{aligned}
d\phi^R_t P^R_{t+\tau}(t, X_t) &= \phi^R_t dP^R_{t+\tau}(t, X_t) + P^R_{t+\tau}(t, X_t) d\phi^R_t + d[\phi^R_t, P^R_{t+\tau}(t, X_t)]_t \\
&= \phi^R_t P\{(-\dot{A}^R(\tau) - \dot{B}^R(\tau)' X_t - B^R(\tau)' K_X X_t \\
&\quad + \frac{1}{2} B^R(\tau)' \Sigma_X \Sigma_X' B^R(\tau))dt + B^R(\tau)' \Sigma_X dZ_t\} \\
&\quad + \phi^R_t P\left(-R_t dt - \Lambda^{R\prime}_t dZ_t\right) - \phi^R_t P B^R(\tau)' \Sigma_X \Lambda^R_t dt \qquad \text{(A.9)} \\
&= \phi^R_t P^R_{t+\tau}(t, X_t)\{(-\dot{A}^R(\tau) - \dot{B}^R(\tau)' X_t - B^R(\tau)' K_X X_t \\
&\quad + \frac{1}{2} B^R(\tau)' \Sigma_X \Sigma_X' B^R(\tau) - R_t - B^R(\tau)' \Sigma_X \Lambda^R_t)dt \\
&\quad + (B^R(\tau)' \Sigma_X - \Lambda^{R\prime}_t) dZ_t\}
\end{aligned}$$

Exploiting the Martingale property of (A.9) leads to the following PDE:

$$\begin{aligned}
0 &= (-\dot{A}^R(\tau) - \dot{B}^R(\tau)' X_t) - B^R(\tau)' K_X X_t + \frac{1}{2} B^R(\tau)' \Sigma_X \Sigma_X' B^R(\tau) \\
&\quad - (R_{0,R} + R_{1,R}' X_t) - B^R(\tau)' \Sigma_X (\Lambda_0 - \sigma_\Pi + \Lambda_1 X_t)
\end{aligned} \qquad \text{(A.10)}$$

This PDE can be split in a stochastic and non stochastic term, leading to the ODE's that describe $A^R(\tau)$ and $B^R(\tau)$:

$$\begin{cases}
\dot{A}^R(\tau) = -R_{0,R} + \frac{1}{2} B^R(\tau)' \Sigma_X \Sigma_X' B^R(\tau) - B^R(\tau)' \Sigma_X (\Lambda_0 - \sigma_\Pi) \\
\dot{B}^R(\tau) = -R_{1,R} - (K_X' + \Lambda_1' \Sigma_X') B^R(\tau)
\end{cases} \qquad \text{(A.11)}$$

Similar to the ODE's describing nominal bond prices, the ODE's in (A.11) obey the following solution:

$$\begin{cases}
A^R(\tau) = \int_0^\tau \dot{A}^R(s) ds \\
B^R(\tau) = (K_X' + \Lambda_1' \Sigma_X')^{-1} \left\{\exp(-(K_X' + \Lambda_1' \Sigma_X')\tau) - \mathrm{I}_{2x2}\right\} R_{1,R}
\end{cases} \qquad \text{(A.12)}$$

Combining (A.3) and (A.11) leads to the price dynamics of an inflation-linked bond:

$$\begin{aligned}
\frac{dP^R_{t+\tau}(t, X_t)}{P^R_{t+\tau}(t, X_t)} &= \{-\dot{A}^R(\tau) - \dot{B}^R(\tau)' X_t - B^R(\tau)' K_X X_t \\
&\quad + \frac{1}{2} B^R(\tau)' \Sigma_X \Sigma_X' B^R(\tau)\} dt + B^R(\tau)' \Sigma_X dZ_t \\
&= \{R_{0,R} - \frac{1}{2} B^R(\tau)' \Sigma_X \Sigma_X' B^R(\tau) + B^R(\tau)' \Sigma_X (\Lambda_0 - \sigma_\Pi) \qquad \text{(A.13)} \\
&\quad - (-R_{1,R} - (K_X' + \Lambda_1' \Sigma_X') B^R(\tau))' X_t - B^R(\tau)' K_X X_t \\
&\quad + \frac{1}{2} B^R(\tau)' \Sigma_X \Sigma_X' B^R(\tau)\} dt + B^R(\tau)' \Sigma_X dZ_t \\
&= \left(R_t + B^R(\tau)' \Sigma_X \Lambda^R_t\right) dt + B^R(\tau)' \Sigma_X dZ_t
\end{aligned}$$

As $X_0 = \Big[0\ 0\Big]'$, we have $P^R_\tau(0, X_0) = \exp\Big(A^R(\tau) + B^R(\tau)'X_0\Big) = \exp(A^R(\tau))$.

Finally, we would like to find the dynamics of $\hat{P}^R_{t+\tau}(t, X_t) = P^R_{t+\tau}(t, X_t)\Pi_t$. An application of Itô's Lemma yields:

$$\begin{aligned}
\frac{d\hat{P}^R_{t+\tau}(t, X_t)}{\hat{P}^R_{t+\tau}(t, X_t)} &= \frac{P^R_{t+\tau}(t, X_t)d\Pi_t}{P^R_{t+\tau}(t, X_t)\Pi_t} + \frac{\Pi_t dP^R_{t+\tau}(t, X_t)}{P^R_{t+\tau}(t, X_t)\Pi_t} + \frac{d[P^R_{t+\tau}(t, X_t), \Pi_t]_t}{P^R_{t+\tau}(t, X_t)\Pi_t} \\
&= \frac{d\Pi_t}{\Pi_t} + \frac{dP^R_{t+\tau}(t, X_t)}{P^R_{t+\tau}(t, X_t)} + B^R(\tau)'\Sigma_X\sigma_\Pi dt \\
&= \Big(R_t + B^R(\tau)'\Sigma_X\Lambda^R_t + \pi_t + B^R(\tau)'\Sigma_X\sigma_\Pi\Big)\, dt \\
&\quad + \Big(B^R(\tau)'\Sigma_X + \sigma'_\Pi\Big)\, dZ_t \\
&= \Big(r_t + B^R(\tau)'\Sigma_X\Lambda_t + \sigma'_\Pi\Lambda_t\Big)\, dt + \Big(B^R(\tau)'\Sigma_X + \sigma'_\Pi\Big)\, dZ_t
\end{aligned} \tag{A.14}$$

A.2 Portfolio optimization problem

To find an explicit representation of $\tilde{P}(t, X_t) = \mathbb{E}\Big[(\frac{\tilde{\phi}^R_T}{\tilde{\phi}^R_t})^{\frac{\gamma-1}{\gamma}}|\mathcal{F}_t\Big]$ we start with deriving the dynamics of $(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}$:

$$\begin{aligned}
d(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}} &= \frac{\gamma-1}{\gamma}(\tilde{\phi}^R_t)^{-\frac{1}{\gamma}}d\tilde{\phi}^R_t - \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\tilde{\phi}^R_t)^{\frac{-\gamma-1}{\gamma}}d[\tilde{\phi}^R_t, \tilde{\phi}^R_t] \\
&= \frac{\gamma-1}{\gamma}(\tilde{\phi}^R_t)^{-\frac{1}{\gamma}}\Big(-\tilde{\phi}^R_t\tilde{R}_t dt - \tilde{\phi}^R_t\tilde{\Lambda}^{R\prime}_t dZ_t\Big) - \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}\tilde{\Lambda}^{R\prime}_t\tilde{\Lambda}^R_t dt \\
&= -(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}(\frac{\gamma-1}{\gamma}\tilde{R}_t + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}^{R\prime}_t\tilde{\Lambda}^R_t)dt - \frac{\gamma-1}{\gamma}(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}\tilde{\Lambda}^{R\prime}_t dZ_t
\end{aligned} \tag{A.15}$$

Then we would like to find an expression for $(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}$, and thus we start by finding the dynamics of $\log(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}$:

$$\begin{aligned}
d\log(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}} &= \frac{1}{(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}}d(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}} - \frac{1}{2}\frac{1}{\Big((\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}\Big)^2}d\Big[(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}, (\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}\Big] \\
&= -(\frac{\gamma-1}{\gamma}\tilde{R}_t + \frac{1}{2}\frac{\gamma-1}{\gamma}\tilde{\Lambda}^{R\prime}_t\tilde{\Lambda}^R_t)dt - \frac{\gamma-1}{\gamma}\tilde{\Lambda}^{R\prime}_t dZ_t
\end{aligned} \tag{A.16}$$

By integrating the SDE above we find the following for $\log(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}}$:

$$\log(\tilde{\phi}^R_t)^{\frac{\gamma-1}{\gamma}} = \int_0^t -(\frac{\gamma-1}{\gamma}\tilde{R}_s + \frac{1}{2}\frac{\gamma-1}{\gamma}\tilde{\Lambda}^{R\prime}_s\tilde{\Lambda}^R_s)ds - \int_0^t \frac{\gamma-1}{\gamma}\tilde{\Lambda}^{R\prime}_s dZ_s \tag{A.17}$$

where we have used that $\tilde{\phi}_0^R = 1$ and therefore $\log(\tilde{\phi}_0^R)^{\frac{\gamma-1}{\gamma}} = 0$. Hence we obtain the subsequent for $(\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}}$:

$$
\begin{aligned}
(\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}} &= \exp\left\{\log(\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}}\right\} \\
&= \exp\left\{\int_0^t -(\frac{\gamma-1}{\gamma}\tilde{R}_s + \frac{1}{2}\frac{\gamma-1}{\gamma}\tilde{\Lambda}_s^{R\prime}\tilde{\Lambda}_s^R)ds - \int_0^t \frac{\gamma-1}{\gamma}\tilde{\Lambda}_s^{R\prime}dZ_s\right\}
\end{aligned}
\tag{A.18}
$$

Consequently, note that $(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}$ reads as follows:

$$
(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}} = \exp\left\{\int_t^T -(\frac{\gamma-1}{\gamma}\tilde{R}_s + \frac{1}{2}\frac{\gamma-1}{\gamma}\tilde{\Lambda}_s^{R\prime}\tilde{\Lambda}_s^R)ds - \int_t^T \frac{\gamma-1}{\gamma}\tilde{\Lambda}_s^{R\prime}dZ_s\right\}
\tag{A.19}
$$

Then, inspired by Kamma and Pelsser (2022), we define the Radon-Nikodym derivative $\xi_t = \frac{dQ}{dP}_{|\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t \lambda_s'\lambda_s ds - \int_0^t \lambda_s' dZ_s\right\}$ where $\lambda_t = \frac{\gamma-1}{\gamma}\tilde{\Lambda}_t^R$. With this Radon-Nikodym derivative we can rewrite our conditional expectation of interest to an expectation under the EMM described by ξ_t:

$$
\begin{aligned}
\tilde{P}(t, X_t) &= \mathbb{E}\left[(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}\middle|\mathcal{F}_t\right] = \mathbb{E}^Q\left[\frac{(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}}{\xi_t}\middle|X_t\right] \\
&= \mathbb{E}^Q\left[\exp\left\{-\int_t^T\left(\frac{\gamma-1}{\gamma}\tilde{R}_s + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_s^{R\prime}\tilde{\Lambda}_s^R\right)ds\right\}\middle|X_t\right]
\end{aligned}
\tag{A.20}
$$

We then define the function $F(t, X_t)$ as follows:

$$
\begin{aligned}
F(t, X_t) &= \frac{\gamma-1}{\gamma}\tilde{R}_t + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_t^{R\prime}\tilde{\Lambda}_t^R \\
&= \frac{\gamma-1}{\gamma}\left(\tilde{R}_{0,R} + \tilde{R}_{1,R}'X_t\right) \\
&\quad + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\tilde{\Lambda}_0 + \tilde{\Lambda}_1 X_t - \sigma_\Pi)'(\tilde{\Lambda}_0 + \tilde{\Lambda}_1 X_t - \sigma_\Pi) \\
&= \frac{\gamma-1}{\gamma}\tilde{R}_{0,R} + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\tilde{\Lambda}_0 - \sigma_\Pi)'(\tilde{\Lambda}_0 - \sigma_\Pi) \\
&\quad + \left(\frac{\gamma-1}{\gamma}\tilde{R}_{1,R} + \frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_1{}'(\tilde{\Lambda}_0 - \sigma_\Pi)\right)' X_t \\
&\quad + \frac{1}{2}\frac{\gamma-1}{\gamma^2}X_t'\tilde{\Lambda}_1'\tilde{\Lambda}_1 X_t
\end{aligned}
\tag{A.21}
$$

We thus see that $\tilde{P}(t, X_t)$ is the conditional expectation of an exponential function that is affine quadratic in the state variables. Conceptually this corresponds to the conditional expectation that Duffie and Kan (1996) solve when deriving the bond prices in an affine yield factor model. Therefore we postulate that $\tilde{P}(t, X_t) = \exp\left(\tilde{A}(t) + \tilde{B}(t)'X_t + X_t'\tilde{C}(t)X_t\right)$, where $\tilde{A}(t) \in \mathbb{R}$,

$\tilde{B}(t) \in \mathbb{R}^{2\times 1}$ and $\tilde{C}(t) \in \mathbb{R}^{2\times 2}$. For a detailed discussion on the derivations of this functional form of the conditional expectation we refer to Chapter 2 and 3 of Duffie and Kan (1996). Applying the Feynman-Kac theorem to our expectation of interest under $\mathbb{Q}$ in (A.20), we find the following PDE that $\tilde{P}(t, X_t)$ has to satisfy:

$$\dot{\tilde{P}} - \tilde{P}_X' K_X X_t + \frac{1}{2} tr(\tilde{P}_{XX}) - (\frac{\gamma-1}{\gamma}\tilde{R}_t + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_t^{R\prime}\tilde{\Lambda}_t^R)\tilde{P} - \frac{\gamma-1}{\gamma}\tilde{P}_X'\Sigma_X\tilde{\Lambda}_t^R = 0 \tag{A.22}$$

If we let $\tilde{P} = \tilde{P}(t, X_t) = \exp\left(\tilde{A}(t) + \tilde{B}(t)'X_t + X_t'\tilde{C}(t)X_t\right)$, we know that its derivatives are as follows:

$$\begin{cases} \dot{\tilde{P}} = \tilde{P}(\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)'X_t + X_t'\dot{\tilde{C}}(t)X_t) \\ \tilde{P}_X = \tilde{P}(\tilde{B}(t) + 2\tilde{C}(t)X_t) \\ \tilde{P}_{XX} = \tilde{P}((\tilde{B}(t) + 2\tilde{C}(t)X_t)(\tilde{B}(t) + 2\tilde{C}(t)X_t)' + 2\tilde{C}(t)) \end{cases} \tag{A.23}$$

Plugging these derivatives in the PDE from (A.22) leads to the following:

$$\begin{aligned} 0 = &\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)'X_t + X_t'\dot{\tilde{C}}(t)X_t - (\tilde{B} + 2\tilde{C}X_t)'K_X X_t \\ &+ \frac{1}{2}\operatorname{tr}\left((\tilde{B}(t) + 2\tilde{C}(t)X_t)(\tilde{B}(t) + 2\tilde{C}(t)X_t)' + 2\tilde{C}(t)\right) \\ &- \left(\frac{\gamma-1}{\gamma}(\tilde{R}_{0,R} + \tilde{R}_{1,R}'X_t) + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\tilde{\Lambda}_0 - \sigma_\Pi + \tilde{\Lambda}_1 X_t)'(\tilde{\Lambda}_0 - \sigma_\Pi + \tilde{\Lambda}_1 X_t)\right) \\ &- \frac{\gamma-1}{\gamma}(\tilde{B}(t) + 2\tilde{C}(t)X_t)'\Sigma_X(\tilde{\Lambda}_0 - \sigma_\Pi + \tilde{\Lambda}_1 X_t) \end{aligned} \tag{A.24}$$

We can separate the PDE above in three parts: a term independent of X_t, a term affine in X_t and a term that is affine quadratic in X_t. Since all three parts should equate to zero, we can simplify the PDE above to a system of ODE's describing $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$:

$$\begin{aligned} \dot{\tilde{A}}(t) = &\frac{\gamma-1}{\gamma}\tilde{B}(t)'\Sigma_X(\tilde{\Lambda}_0 - \sigma_\Pi) - \frac{1}{2}\tilde{B}(t)'\tilde{B}(t) - \operatorname{tr}(\tilde{C}(t)) + \frac{\gamma-1}{\gamma}\tilde{R}_{0,R} \\ &+ \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\tilde{\Lambda}_0 - \sigma_\Pi)'(\tilde{\Lambda}_0 - \sigma_\Pi) \\ \dot{\tilde{B}}(t) = &\left(\frac{\gamma-1}{\gamma}\tilde{\Lambda}_1'\Sigma_X' + K_X' - 2\tilde{C}(t)'\right)\tilde{B}(t) + 2\frac{\gamma-1}{\gamma}\tilde{C}(t)'\Sigma_X(\tilde{\Lambda}_0 - \sigma_\Pi) \\ &+ \frac{\gamma-1}{\gamma}\tilde{R}_{1,R} + \frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_1'(\tilde{\Lambda}_0 - \sigma_\Pi) \\ \dot{\tilde{C}}(t) = &2\left(K_X' + \frac{\gamma-1}{\gamma}\tilde{\Lambda}_1'\Sigma_X'\right)\tilde{C}(t) - 2\tilde{C}(t)'\tilde{C}(t) + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_1'\tilde{\Lambda}_1 \end{aligned} \tag{A.25}$$

By combining (2.29) and (2.30) we know that $\mathbb{E}\left[(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_T^R})^{\frac{\gamma-1}{\gamma}}|\mathcal{F}_t\right] = 1$ and thus we find the terminal conditions $\tilde{A}(T) = 0$, $\tilde{B}(T) = 0_{2\times 1}$ and $\tilde{C}(T) = 0_{2\times 2}$. Note that no closed form solution for the system is available.

To derive the dynamics of $\tilde{W}_t^* \tilde{\phi}_t^N = \frac{W_0}{g_T}(\tilde{\phi}_t^R)^{1-\frac{1}{\gamma}} \tilde{P}(t, X_t)$, we first derive the dynamics of $\tilde{P}(t, X_t) = \exp\left(\tilde{A}(t) + \tilde{B}(t)' X_t + X_t' \tilde{C}(t) X_t\right)$:

$$\begin{aligned} d\tilde{P}(t, X_t) &= \dot{\tilde{P}} dt + \tilde{P}_X dX_t + \frac{1}{2} \tilde{P}_{XX} d[X, X]_t \\ &= \tilde{P}(t, X_t)\{\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)' X_t + X_t' \dot{\tilde{C}} X_t \\ &\quad + \frac{1}{2} \operatorname{tr}\left((\tilde{B}(t) + 2\tilde{C}(t) X_t)(\tilde{B}(t) + 2\tilde{C} X_t)' + 2\tilde{C}(t)\right) \\ &\quad - (\tilde{B}(t) + 2\tilde{C}(t) X_t)' K_X X_t\} dt + \tilde{P}(t, X_t)(\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma_X dZ_t \end{aligned} \tag{A.26}$$

where we have used that $\Sigma_X \Sigma_X' = I_{2\times2}$. Then by combining (A.15) and (A.26), we find the following dynamics of $\tilde{W}_t^* \tilde{\phi}_t^N$:

$$\begin{aligned} d\tilde{W}_t^* \tilde{\phi}_t^N &= d\frac{W_0}{g_T}(\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}} \tilde{P}(t, X_t) \\ &= \frac{W_0}{g_T}\left((\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}} d\tilde{P}(t, X_t) + \tilde{P}(t, X_t) d(\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}} + d\left[(\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}}, \tilde{P}(t, X_t)\right]\right) \\ &= \frac{W_0}{g_T}(\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}} \tilde{P}(t, X_t)\Big(\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)' X_t + X_t' \dot{\tilde{C}}(t) X_t \\ &\quad - (\tilde{B}(t) + 2\tilde{C}(t) X_t)' K_X X_t + \frac{1}{2} \operatorname{tr}\Big((\tilde{B}(t) + 2\tilde{C}(t) X_t)(\tilde{B}(t) + 2\tilde{C}(t) X_t)' \\ &\quad + 2\tilde{C}(t)\Big) - \Big(\frac{\gamma-1}{\gamma} \tilde{R}_t + \frac{1}{2}\frac{\gamma-1}{\gamma^2} \tilde{\Lambda}_t^{R\prime} \tilde{\Lambda}_t^R\Big) \\ &\quad - \frac{\gamma-1}{\gamma}(\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma_X \tilde{\Lambda}_t^R\Big) dt \\ &\quad + \left((\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma_X - \frac{\gamma-1}{\gamma} \tilde{\Lambda}_t^{R\prime}\right) dZ_t \\ &= \tilde{W}_t^* \tilde{\phi}_t^N \left((\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma_X - \frac{\gamma-1}{\gamma} \tilde{\Lambda}_t^{R\prime}\right) dZ_t \end{aligned} \tag{A.27}$$

where we have used (A.24) to cancel the drift term and the fact that $\tilde{W}_t^* \tilde{\phi}_t^N = \frac{W_0}{g_T}(\tilde{\phi}_t^R)^{\frac{\gamma-1}{\gamma}} \tilde{P}(t, X_t)$.

Then we find a second way to describe $\tilde{W}_t^* \tilde{\phi}_t^N$, based on the dynamic budget constraint. Note that the budget constraint reads as follows:

$$d\tilde{W}_t^* = \tilde{W}_t^*(r_t + \tilde{\theta}_t^*(a)' \Sigma \tilde{\Lambda}_t) dt + \tilde{W}_t^* \tilde{\theta}_t^*(a)' \Sigma dZ_t \tag{A.28}$$

Hence, we find the following alternative dynamics of $\tilde{W}_t^* \tilde{\phi}_t^N$:

$$
\begin{aligned}
d\tilde{W}_t^* \tilde{\phi}_t^N &= \tilde{W}_t^* d\tilde{\phi}_t^N + \tilde{\phi}_t^N d\tilde{W}_t^* + d[\tilde{W}_t^*, \tilde{\phi}_t^N] \\
&= \tilde{W}_t^* \tilde{\phi}_t^N (-r_t dt - \tilde{\Lambda}_t' dZ_t) + \tilde{W}_t^* \tilde{\phi}_t^N \Big((r_t + \tilde{\theta}_t^*(a)' \Sigma \tilde{\Lambda}_t) dt + \tilde{\theta}_t^*(a)' \Sigma dZ_t \Big) \\
&\quad - \tilde{W}_t^* \tilde{\phi}_t^N \tilde{\theta}_t^*(a)' \Sigma \tilde{\Lambda}_t dt \\
&= \tilde{W}_t^* \tilde{\phi}_t^N (\tilde{\theta}_t^*(a)' \Sigma - \tilde{\Lambda}_t') dZ_t
\end{aligned}
\tag{A.29}
$$

By equating the volatility terms in (A.27) and (A.29), we arrive at the following equation that defines the optimal investment strategy in the dual market:

$$
(\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma_X - \frac{\gamma - 1}{\gamma} \tilde{\Lambda}_t^{R\prime} = \tilde{\theta}_t^*(a)' \Sigma - \tilde{\Lambda}_t' \tag{A.30}
$$

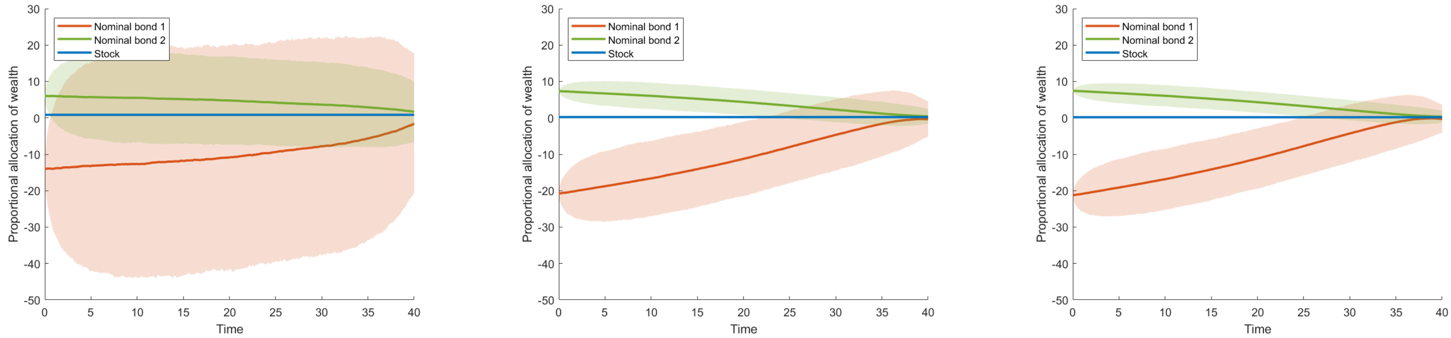
We therefore have the following optimal strategy in the dual market:

$$
\begin{aligned}
\tilde{\theta}_t^*(a)' &= \left(\left(\begin{bmatrix} \tilde{B}_{1,t} \\ \tilde{B}_{2,t} \end{bmatrix} + \begin{bmatrix} 2\tilde{C}_{11,t} & 2\tilde{C}_{12,t} \\ 2\tilde{C}_{21,t} & 2\tilde{C}_{22,t} \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} \right)' [I_{2\times 2} \quad 0_{2\times 2}] - \frac{\gamma-1}{\gamma} \tilde{\Lambda}_t^{R\prime} + \tilde{\Lambda}_t' \right) \Sigma^{-1} \\
&= \left(\begin{bmatrix} \tilde{B}_{1,t} + 2\tilde{C}_{11,t} X_{1,t} + 2\tilde{C}_{12,t} X_{2,t} \\ \tilde{B}_{2,t} + 2\tilde{C}_{21,t} X_{1,t} + 2\tilde{C}_{22,t} X_{2,t} \\ 0 \\ 0 \end{bmatrix}' + \frac{1}{\gamma} \tilde{\Lambda}_t' + \frac{\gamma - 1}{\gamma} \sigma_\Pi' \right) \Sigma^{-1} \\
&= \frac{1}{\gamma} \tilde{\Lambda}_t' \Sigma^{-1} + \frac{\gamma - 1}{\gamma} \sigma_\Pi' \Sigma^{-1} + \tilde{H}(t, X_t)' \Sigma^{-1}
\end{aligned}
\tag{A.31}
$$

where $\tilde{H}(t, X_t) = \begin{bmatrix} \tilde{B}_{1,t} + 2\tilde{C}_{11,t} X_{1,t} + 2\tilde{C}_{12,t} X_{2,t} \\ \tilde{B}_{2,t} + 2\tilde{C}_{21,t} X_{1,t} + 2\tilde{C}_{22,t} X_{2,t} \\ 0 \\ 0 \end{bmatrix}$. Taking the transpose of all entities in (A.31) leads to optimal strategy in the dual market:

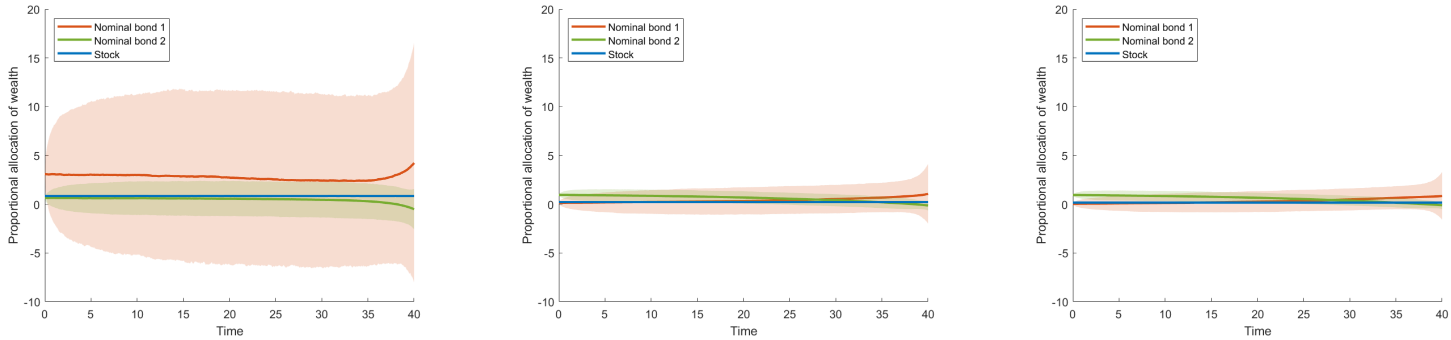
$$
\tilde{\theta}_t^*(a)' = \frac{1}{\gamma} \left(\Sigma^{-1} \right)^\top \tilde{\Lambda}_t + \frac{\gamma - 1}{\gamma} \left(\Sigma^{-1} \right)^\top \sigma_\Pi + \left(\Sigma^{-1} \right)^\top \tilde{H}(t, X_t) \tag{A.32}
$$

A.3 Numerical implementation investment strategy



(a) $\gamma = 2$ (b) $\gamma = 8$ (c) $\gamma = 10$

Figure A.1: Comparison of the optimal strategy in the KNW model between different values of γ for the alternative parameters. The average allocations to the stock and two nominal bonds over 10,000 simulations of the optimal investment strategy are shown in thick lines. The 90% percentile range is shown in the shaded areas. We assume $T = 40$ with monthly time steps in the simulations. Furthermore, $\tau_1 = 1$, $\tau_2 = 5$, $\tau = 5$, and $W_0 = 1$. Other parameter values are given in Table 2.2.



(a) $\gamma = 2$ (b) $\gamma = 8$ (c) $\gamma = 10$

Figure A.2: Comparison of the optimal strategy in the KNW model between different values of γ for the baseline parameters. The average allocations to the stock and two nominal bonds over $10,000$ simulations of the optimal investment strategy are shown in thick lines. The 90% percentile range is shown in the shaded areas. We assume $T = 40$ with monthly time steps in the simulations. Furthermore, $\tau_1 = 5$, $\tau_2 = 20$, $\tau = 10$, and $W_0 = 1$. Other parameter values are given in Table 2.1.

A.4 Welfare analysis

γ	2	3	4	5	6	7	8	9	10
CE KNW baseline	57.82	18.49	10.60	7.59	6.06	5.16	4.56	4.15	3.84
	(3.50)	(1.26)	(0.66)	(0.42)	(0.30)	(0.22)	(0.18)	(0.15)	(0.12)
CE myopic baseline	45.36	13.29	7.40	5.19	4.05	3.37	2.90	2.56	2.31
	(2.24)	(0.63)	(0.28)	(0.16)	(0.11)	(0.08)	(0.07)	(0.07)	(0.07)
CE static baseline	5.33	3.29	2.57	2.19	1.94	1.76	1.61	1.50	1.40
	(0.12)	(0.07)	(0.06)	(0.05)	(0.04)	(0.04)	(0.03)	(0.03)	(0.03)
CE KNW alternative	9.34	4.92	3.59	2.98	2.63	2.41	2.25	2.14	2.05
	(0.28)	(0.18)	(0.14)	(0.11)	(0.09)	(0.08)	(0.07)	(0.06)	(0.05)
CE myopic alternative	8.32	4.04	2.73	2.08	1.68	1.41	1.21	1.06	0.95
	(0.21)	(0.10)	(0.07)	(0.06)	(0.06)	(0.06)	(0.06)	(0.05)	(0.05)
CE static alternative	3.04	2.11	1.66	1.37	1.15	0.98	0.84	0.75	0.67
	(0.04)	(0.03)	(0.03)	(0.04)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)

Table A.1: Rounded certainty equivalents underlying the welfare losses presented in Table 2.3. The baseline parameters can be found in Table 2.1, the alternative parameters in Table 2.2. Standard errors are reported in parentheses. The data is calculated over $10,000$ simulations. We assume $T = 40$ with monthly time steps in the simulations, $W_0 = 1$, $\tau_1 = 1$, $\tau_2 = 5$, and $\tau = 5$.

True γ \ Measured γ	2	3	4	5	6	7	8	9	10
2	–	46.34	29.58	20.26	15.04	11.89	9.84	8.44	7.42
		(1.07)	(0.42)	(0.21)	(0.12)	(0.08)	(0.06)	(0.04)	(0.04)
3	9.42	–	17.11	14.07	11.58	9.76	8.44	7.46	6.71
	(1.80)		(0.50)	(0.23)	(0.13)	(0.08)	(0.06)	(0.04)	(0.03)
4	2.99	8.70	–	10.17	9.16	8.17	7.33	6.66	6.11
	(0.64)	(1.11)		(0.33)	(0.17)	(0.10)	(0.07)	(0.05)	(0.04)
5	1.52	5.00	7.05	–	7.38	6.93	6.44	5.99	5.59
	(0.29)	(0.73)	(0.66)		(0.24)	(0.14)	(0.09)	(0.06)	(0.05)
6	1.00	3.40	5.11	5.88	–	5.94	5.69	5.42	5.15
	(0.17)	(0.48)	(0.55)	(0.44)		(0.19)	(0.12)	(0.08)	(0.06)
7	0.75	2.58	3.99	4.76	5.10	–	5.08	4.93	4.75
	(0.11)	(0.34)	(0.43)	(0.40)	(0.31)		(0.16)	(0.11)	(0.08)
8	0.60	2.11	3.30	4.01	4.39	4.55	–	4.51	4.41
	(0.08)	(0.25)	(0.34)	(0.34)	(0.30)	(0.24)		(0.13)	(0.10)
9	0.52	1.81	2.85	3.50	3.87	4.07	4.15	–	4.10
	(0.06)	(0.20)	(0.28)	(0.29)	(0.27)	(0.23)	(0.19)		(0.11)
10	0.46	1.60	2.53	3.13	3.49	3.70	3.81	3.85	–
	(0.05)	(0.16)	(0.23)	(0.25)	(0.24)	(0.22)	(0.19)	(0.15)	

Table A.2: Rounded certainty equivalents when an agent invests according to the measured γ, whereas the utility is evaluated on the basis of the true γ. The welfare losses in Table 2.4 can be calculated when combining this table with the certainty equivalents for the baseline KNW strategy in Table A.1. The data is calculated over $10,000$ simulations. We assume $T = 40$ with monthly time steps in the simulations, $W_0 = 1$, $\tau_1 = 1$, $\tau_2 = 5$, and $\tau = 5$. Other parameter values are given in Table 2.1.

True γ \ Measured γ	2	3	4	5	6	7	8	9	10
2	–	8.08	6.30	5.13	4.37	3.85	3.48	3.20	2.99
		(0.11)	(0.06)	(0.03)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)
3	2.96	–	4.74	4.26	3.82	3.48	3.22	3.01	2.84
	(0.48)		(0.07)	(0.04)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)
4	1.03	2.94	–	3.56	3.38	3.17	2.99	2.84	2.71
	(0.24)	(0.31)		(0.06)	(0.03)	(0.02)	(0.02)	(0.01)	(0.01)
5	0.52	1.80	2.69	–	2.98	2.90	2.79	2.68	2.58
	(0.11)	(0.27)	(0.21)		(0.06)	(0.03)	(0.02)	(0.02)	(0.01)
6	0.33	1.23	2.03	2.47	–	2.64	2.60	2.54	2.47
	(0.06)	(0.19)	(0.22)	(0.16)		(0.05)	(0.03)	(0.02)	(0.02)
7	0.25	0.93	1.59	2.06	2.31	–	2.42	2.40	2.36
	(0.04)	(0.13)	(0.19)	(0.17)	(0.12)		(0.05)	(0.03)	(0.02)
8	0.20	0.75	1.32	1.75	2.03	2.19	–	2.27	2.26
	(0.03)	(0.10)	(0.15)	(0.16)	(0.14)	(0.10)		(0.04)	(0.03)
9	0.17	0.64	1.13	1.52	1.80	1.99	2.09	–	2.15
	(0.02)	(0.07)	(0.12)	(0.14)	(0.14)	(0.11)	(0.09)		(0.04)
10	0.15	0.57	1.00	1.36	1.63	1.82	1.94	2.01	–
	(0.02)	(0.06)	(0.10)	(0.12)	(0.13)	(0.12)	(0.10)	(0.07)	

Table A.3: Rounded certainty equivalents when an agent invests according to the measured γ, whereas the utility is evaluated on the basis of the true γ. The welfare losses in Table 2.5 can be calculated when combining this table with the certainty equivalents for the alternative KNW strategy in Table A.1. The data is calculated over $10,000$ simulations. We assume $T = 40$ with monthly time steps in the simulations, $W_0 = 1$, $\tau_1 = 1$, $\tau_2 = 5$, and $\tau = 5$. Other parameter values are given in Table 2.2.

Appendix B

Stochastic volatility

B.1 Financial market

To find the dynamics of $O_t = h(t, S_t, \nu_t)$, we first apply Itô's Lemma to $h(t, S_t, \nu_t)$:

$$
\begin{aligned}
dO_t &= \dot{h}dt + h_S dS_t + h_\nu d\nu_t + \frac{1}{2}h_{SS}d[S,S]_t + \frac{1}{2}h_{\nu\nu}d[\nu,\nu]_t + h_{S\nu}d[S,\nu]_t \\
&= \dot{h}dt + h_S(\mu_t S_t dt + \sqrt{\nu_t}S_t dZ_t^1) \\
&\quad + h_\nu(\kappa(\bar{\nu} - \nu_t)dt + \delta\sqrt{\nu_t}(\rho dZ_t^1 + \sqrt{1-\rho^2}dZ_t^2)) + \frac{1}{2}h_{SS}\nu_t S_t^2 dt \\
&\quad + \frac{1}{2}h_{\nu\nu}\delta^2\nu_t dt + h_{S\nu}\delta\nu_t S_t\rho dt \\
&= \left(\dot{h} + h_S\mu_t S_t + h_\nu\kappa(\bar{\nu} - \nu_t) + \frac{1}{2}\nu_t h_{SS}S_t^2 + \frac{1}{2}\nu_t h_{\nu\nu}\delta^2 + h_{S\nu}\delta\nu_t S_t\rho\right)dt \\
&\quad + h_S\sqrt{\nu_t}S_t dZ_t^1 + h_\nu\delta\sqrt{\nu_t}(\rho dZ_t^1 + \sqrt{1-\rho^2}dZ_t^2)
\end{aligned} \tag{B.1}
$$

We can simplify (B.1) by using the martingale property of $\phi_t O_t$. Hence, we first have to find the dynamics of $\phi_t O_t$:

$$
\begin{aligned}
d\phi_t O_t &= \phi_t dO_t + O_t d\phi_t + d[\phi, O]_t \\
&= \phi_t\bigg(\Big(\dot{h} + h_S \mu_t S_t + h_\nu \kappa(\bar{\nu} - \nu_t) + \frac{1}{2}\nu_t h_{SS} S_t^2 + \frac{1}{2}\nu_t h_{\nu\nu}\delta^2 + h_{S\nu}\delta\nu_t S_t \rho\Big)dt \\
&\quad + h_S\sqrt{\nu_t} S_t dZ_t^1 + h_\nu \delta\sqrt{\nu_t}(\rho dZ_t^1 + \sqrt{1-\rho^2} dZ_t^2)\bigg) \\
&\quad + O_t\Big(- \phi_t r dt - \phi_t(\eta_1\sqrt{\nu_t} dZ_t^1 + \eta_2\sqrt{\nu_t} dZ_t^2)\Big) \\
&\quad - \phi_t\Big((h_S S_t + h_\nu\delta\rho)\eta_1\nu_t + h_\nu\delta\sqrt{1-\rho^2}\eta_2\nu_t\Big)dt \\
&= \phi_t\Big(\dot{h} + h_S \mu_t S_t + h_\nu \kappa(\bar{\nu} - \nu_t) + \frac{1}{2}\nu_t h_{SS} S_t^2 + \frac{1}{2}\nu_t h_{\nu\nu}\delta^2 + h_{S\nu}\delta\nu_t S_t \rho \\
&\quad - O_t r - (h_S S_t + h_\nu\delta\rho)\eta_1\nu_t - h_\nu\delta\sqrt{1-\rho^2}\eta_2\nu_t\Big)dt \\
&\quad + \phi_t\Big(h_S\sqrt{\nu_t} S_t dZ_t^1 + h_\nu\delta\sqrt{\nu_t}(\rho dZ_t^1 + \sqrt{1-\rho^2} dZ_t^2) \\
&\quad - O_t(\eta_1\sqrt{\nu_t} dZ_t^1 + \eta_2\sqrt{\nu_t} dZ_t^2)\Big)
\end{aligned}
\tag{B.2}
$$

Since $\phi_t O_t$ should be a martingale by FTAP, we know that its drift term should equate to zero. Using this, we can simplify (B.1) to the following:

$$
\begin{aligned}
dO_t = &\Big(rO_t + (h_S S_t + h_\nu\delta\rho)\eta_1\nu_t + h_\nu\delta\sqrt{1-\rho^2}\eta_2\nu_t\Big)dt \\
&+ \sqrt{\nu_t}\Big((h_S S_t + h_\nu\delta\rho)dZ_t^1 + h_\nu\delta\sqrt{1-\rho^2} dZ_t^2\Big)
\end{aligned}
\tag{B.3}
$$

B.2 Portfolio optimization problem

To derive an explicit expression for $\tilde{P}(t, \nu_t) = \mathbb{E}\left[\left(\frac{\tilde{\phi}_T}{\tilde{\phi}_t}\right)^{\frac{\gamma-1}{\gamma}} \middle| \mathcal{F}_t\right]$ we start with deriving the dynamics of $(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}$:

$$
\begin{aligned}
d(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}} &= \frac{\gamma-1}{\gamma}(\tilde{\phi}_t)^{-\frac{1}{\gamma}} d\tilde{\phi}_t - \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\tilde{\phi}_t)^{\frac{-\gamma-1}{\gamma}} d[\tilde{\phi}_t, \tilde{\phi}_t] \\
&= \frac{\gamma-1}{\gamma}(\tilde{\phi}_t)^{-\frac{1}{\gamma}}\Big(- \tilde{\phi}_t r dt - \tilde{\phi}_t(\eta_1\sqrt{\nu_t} dZ_t^1 + \tilde{\eta}_{2,t}\sqrt{\nu_t} dZ_t^2)\Big) \\
&\quad - \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}(\eta_1^2 + \tilde{\eta}_{2,t}^2)\nu_t dt \\
&= -(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}\Big(\frac{\gamma-1}{\gamma} r + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\eta_1^2 + \tilde{\eta}_{2,t}^2)\nu_t\Big)dt \\
&\quad - (\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}\frac{\gamma-1}{\gamma}\Big(\eta_1\sqrt{\nu_t} dZ_t^1 + \tilde{\eta}_{2,t}\sqrt{\nu_t} dZ_t^2\Big)
\end{aligned}
\tag{B.4}
$$

Similar to Appendix A.2 we would like to find an expression for $(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}$. We can do so by finding the dynamics of $\log(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}$, integrating these dynamics and taking the exponential of the last found equation. As this procedure coincides with the steps taken in Appendix A.2 we choose to not show the step by step derivations. We find that $(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}$ reads as follows:

$$(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}} = \exp\left\{\int_0^t -\Big(\frac{\gamma-1}{\gamma}r + \frac{1}{2}\frac{\gamma-1}{\gamma}(\eta_1^2 + \tilde{\eta}_{2,s}^2)\nu_s\Big)ds - \frac{\gamma-1}{\gamma}\int_0^t \tilde{\Lambda}_s^{'} dZ_s\right\} \tag{B.5}$$

Consequently, note that $(\frac{\tilde{\phi}_T}{\tilde{\phi}_t})^{\frac{\gamma-1}{\gamma}}$ reads as follows:

$$(\frac{\tilde{\phi}_T}{\tilde{\phi}_t})^{\frac{\gamma-1}{\gamma}} = \exp\left\{\int_t^T -\Big(\frac{\gamma-1}{\gamma}r + \frac{1}{2}\frac{\gamma-1}{\gamma}(\eta_1^2 + \tilde{\eta}_{2,s}^2)\nu_s\Big)ds - \frac{\gamma-1}{\gamma}\int_t^T \tilde{\Lambda}_s^{'} dZ_s\right\} \tag{B.6}$$

To show that $\tilde{P}(t,\nu_t)$ is the conditional expectation under a specific measure of an exponentially affine function of the stochastic variance, we define the Radon-Nikodym derivative $\xi_t = \frac{dQ}{dP}_{|\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t \lambda_s^{\prime}\lambda_s ds - \int_0^t \lambda_s^{\prime} dZ_s\right\}$ where $\lambda_t = \frac{\gamma-1}{\gamma}\tilde{\Lambda}_t$. With this Radon-Nikodym derivative we can rewrite our conditional expectation of interest to an expectation under the EMM described by ξ_t:

$$\begin{aligned}\tilde{P}(t,X_t) &= \mathbb{E}\left[(\frac{\tilde{\phi}_T}{\tilde{\phi}_t})^{\frac{\gamma-1}{\gamma}}\middle|\mathcal{F}_t\right] = \mathbb{E}^Q\left[\frac{(\frac{\tilde{\phi}_T}{\tilde{\phi}_t})^{\frac{\gamma-1}{\gamma}}}{\xi_t}\middle|\nu_t\right] \\ &= \mathbb{E}^Q\left[\exp\left\{-\int_t^T\left(\frac{\gamma-1}{\gamma}r + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\eta_1^2 + \tilde{\eta}_{2,s}^2)\nu_s\right)ds\right\}\middle|\nu_t\right]\end{aligned} \tag{B.7}$$

Hence, we see that $\tilde{P}(t,\nu_t)$ is an exponentially affine function of the stochastic variance ν_t. Since the dynamics of ν_t in (3.2) are described by a univariate Markov diffusion process, our conditional expectation of interest is conceptually the same as the conditional expectation that Duffie and Kan (1996) solve when determining yield curves. Therefore we postulate that $\tilde{P}(t,\nu_t) = \exp\Big(\tilde{A}(t) + \tilde{B}(t)\nu_t\Big)$, where $\tilde{A}(t) \in \mathbb{R}$ and $\tilde{B}(t) \in \mathbb{R}$. Note that without the assumption that the prices of risk are proportional to the square root of the stochastic variance process, the conditional expectation under $\mathbb{Q}$ above would not have been affine in the state variables. Consequently, it would not be possible to find an analytical expression for our conditional expectation of interest without this assumption. As a consequence of the results by Duffie and Kan (1996), we can apply the Feynman-Kac theorem to our expectation under $\mathbb{Q}$ in (B.7). We then find the following PDE that describes $\tilde{A}(t)$ and

$\tilde{B}(t)$:

$$\begin{aligned}0 = \dot{\tilde{P}} + \tilde{P}_\nu \kappa(\bar{\nu} - \nu_t) + \frac{1}{2}\tilde{P}_{\nu\nu}\delta^2\nu_t - (\frac{\gamma-1}{\gamma}r + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\eta_1^2 + \tilde{\eta}_{2,t}^2)\nu_t)\tilde{P} \\ - \frac{\gamma-1}{\gamma}\delta\nu_t(\rho\eta_1 + \sqrt{1-\rho^2}\tilde{\eta}_{2,t})\tilde{P}_\nu\end{aligned} \tag{B.8}$$

We know that the derivatives of $\tilde{P}(t, \nu_t)$ look as follows:

$$\begin{cases} \dot{\tilde{P}} = \tilde{P}(t,\nu_t)\Big(\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)\nu_t\Big) \\ \tilde{P}_\nu = \tilde{P}(t,\nu_t)\tilde{B}(t) \\ \tilde{P}_{\nu\nu} = \tilde{P}(t,\nu_t)\tilde{B}(t)^2 \end{cases} \tag{B.9}$$

Plugging these derivatives in the PDE from (B.8) and separating the terms multiplied by ν_t from the terms not multiplied by ν_t, we find the following two ODE's that describe $\tilde{A}(t)$ and $\tilde{B}(t)$:

$$\begin{cases} \dot{\tilde{A}}(t) = -\kappa\bar{\nu}\tilde{B}(t) + \dfrac{\gamma-1}{\gamma}r \\ \dot{\tilde{B}}(t) = -\dfrac{1}{2}\delta^2\tilde{B}(t)^2 + \Big(\kappa + \dfrac{\gamma-1}{\gamma}\delta(\rho\eta_1 + \sqrt{1-\rho^2}\tilde{\eta}_{2,t})\Big)\tilde{B}(t) \\ \qquad + \dfrac{1}{2}\dfrac{\gamma-1}{\gamma^2}(\eta_1^2 + \tilde{\eta}_{2,t}^2) \end{cases} \tag{B.10}$$

By combining (3.18) and (3.19) we know that $\mathbb{E}\Big[(\frac{\tilde{\phi}_T}{\tilde{\phi}_t})^{\frac{\gamma-1}{\gamma}}\Big|\mathcal{F}_t\Big] = 1$ and thus we find the terminal conditions $\tilde{A}(T) = \tilde{B}(T) = 0$. $\tilde{A}(t)$ and $\tilde{B}(t)$ have a closed form solution which we present in Proposition 3.1 and (3.28), respectively.

To derive the dynamics of $\tilde{W}_t^* \tilde{\phi}_t = \frac{W_0}{g_T}(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}\tilde{P}(t,\nu_t)$, we first derive the dynamics of $\tilde{P}(t,\nu_t)$:

$$\begin{aligned} d\tilde{P}(t,\nu_t) &= \dot{\tilde{P}}dt + \tilde{P}_\nu d\nu_t + \frac{1}{2}\tilde{P}_{\nu\nu}d[\nu,\nu]_t \\ &= \tilde{P}(t,\nu_t)\Big(\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)\nu_t\Big)dt \\ &\quad + \tilde{P}(t,\nu_t)\tilde{B}(t)\Big(\kappa(\bar{\nu}-\nu_t)dt + \delta\sqrt{\nu_t}(\rho dZ_t^1 + \sqrt{1-\rho^2}dZ_t^2)\Big) \\ &\quad + \frac{1}{2}\tilde{P}(t,\nu_t)\tilde{B}(t)^2\delta^2\nu_t \\ &= \tilde{P}(t,\nu_t)\Big(\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)\nu_t + \tilde{B}(t)\kappa(\bar{\nu}-\nu_t) + \frac{1}{2}\tilde{B}(t)^2\delta^2\nu_t\Big)dt \\ &\quad + \tilde{P}(t,\nu_t)\tilde{B}(t)\delta\sqrt{\nu_t}(\rho dZ_t^1 + \sqrt{1-\rho^2}dZ_t^2) \end{aligned} \tag{B.11}$$

Then by using (B.4) and (B.11) we can find the dynamics of $\tilde{W}_t^*\tilde{\phi}_t$:

$$\begin{aligned}
d\tilde{W}_t^*\tilde{\phi}_t &= d\frac{W_0}{g_T}(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}\tilde{P}(t,\nu_t) \\
&= \frac{W_0}{g_T}\Big((\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}d\tilde{P}(t,\nu_t) + \tilde{P}(t,\nu_t)d(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}} + d\Big[(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}, \tilde{P}(t,\nu_t)\Big]\Big) \\
&= \frac{W_0}{g_T}(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}\tilde{P}(t,\nu_t)\Bigg(\Big(\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)\nu_t + \tilde{B}(t)\kappa(\bar{\nu}-\nu_t) + \frac{1}{2}\tilde{B}(t)^2\delta^2\nu_t \\
&\quad - \frac{\gamma-1}{\gamma}r - \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\eta_1^2 + \tilde{\eta}_{2,t}^2)\nu_t - \frac{\gamma-1}{\gamma}\delta\nu_t(\rho\eta_1 + \sqrt{1-\rho^2}\tilde{\eta}_{2,t})\Big)dt \\
&\quad + \Big(\tilde{B}(t)\delta\rho - \frac{\gamma-1}{\gamma}\eta_1\Big)\sqrt{\nu_t}dZ_t^1 + \Big(\tilde{B}(t)\delta\sqrt{1-\rho^2} - \frac{\gamma-1}{\gamma}\tilde{\eta}_{2,t}\Big)\sqrt{\nu_t}dZ_t^2\Bigg) \\
&= \tilde{W}_t^*\tilde{\phi}_t\Big((\tilde{B}(t)\delta\rho - \frac{\gamma-1}{\gamma}\eta_1)\sqrt{\nu_t}dZ_t^1 \\
&\quad + (\tilde{B}(t)\delta\sqrt{1-\rho^2} - \frac{\gamma-1}{\gamma}\tilde{\eta}_{2,t})\sqrt{\nu_t}dZ_t^2\Big)
\end{aligned} \tag{B.12}$$

where we have used (B.8) to cancel the drift term and the fact that $\tilde{W}_t^*\tilde{\phi}_t = \frac{W_0}{g_T}(\tilde{\phi}_t)^{\frac{\gamma-1}{\gamma}}\tilde{P}(t,\nu_t)$.

Then we find a second way to describe $\tilde{W}_t^*\tilde{\phi}_t$, based on the dynamic budget constraint. Note that the budget constraint in our dual market reads as follows:

$$d\tilde{W}_t^* = \tilde{W}_t^*(r + \tilde{\theta}_t^*(a)'\Sigma_t\tilde{\Lambda}_t)dt + \tilde{W}_t^*\tilde{\theta}_t^*(a)'\Sigma_t dZ_t \tag{B.13}$$

Hence we find the following alternative dynamics of $\tilde{W}_t^*\tilde{\phi}_t$:

$$\begin{aligned}
d\tilde{W}_t^*\tilde{\phi}_t &= \tilde{W}_t^*d\tilde{\phi}_t + \tilde{\phi}_t d\tilde{W}_t^* + d[\tilde{W}_t^*, \tilde{\phi}_t] \\
&= \tilde{W}_t^*\tilde{\phi}_t(-rdt - \tilde{\Lambda}_t'dZ_t) + \tilde{W}_t^*\tilde{\phi}_t\Big((r + \tilde{\theta}_t^*(a)'\Sigma_t\tilde{\Lambda}_t)dt + \tilde{\theta}_t^*(a)'\Sigma_t dZ_t\Big) \\
&\quad - \tilde{W}_t^*\tilde{\phi}_t\tilde{\theta}_t^*(a)''\Sigma_t\tilde{\Lambda}_t dt \\
&= \tilde{W}_t^*\tilde{\phi}_t(\tilde{\theta}_t^*(a)'\Sigma_t - \tilde{\Lambda}_t')dZ_t
\end{aligned} \tag{B.14}$$

By equating the volatility terms in (B.12) and (B.14) we find an equation that describes the optimal investment strategy in the dual market:

$$\Big[\tilde{B}(t)\delta\rho - \tfrac{\gamma-1}{\gamma}\eta_1 \quad \tilde{B}(t)\delta\sqrt{1-\rho^2} - \tfrac{\gamma-1}{\gamma}\tilde{\eta}_{2,t}\Big]\sqrt{\nu_t} = \tilde{\theta}_t^*(a)'\Sigma_t - \tilde{\Lambda}_t' \tag{B.15}$$

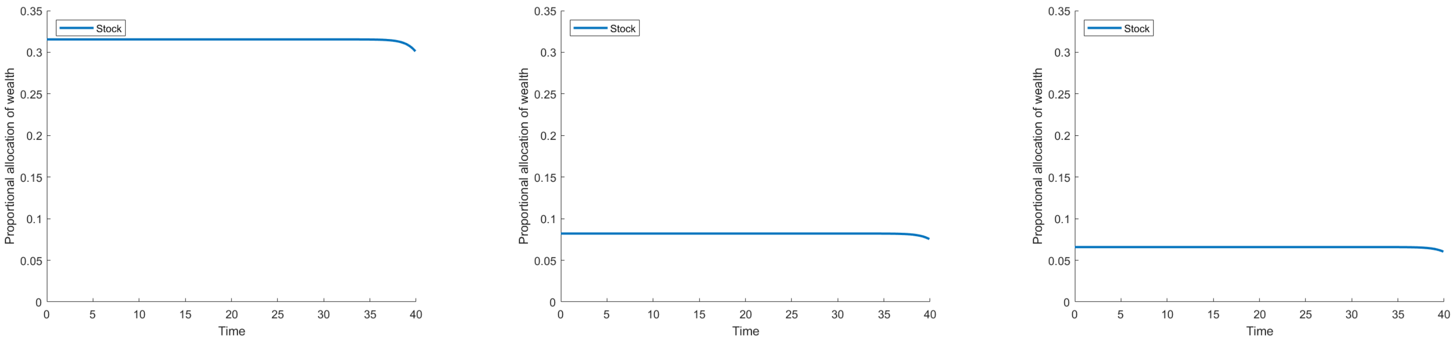
We therefore have the following system with two unknowns, $\tilde{\theta}_{1,t}^*(a)$ and $\tilde{\theta}_{2,t}^*(a)$:

$$\begin{cases}
(\tilde{B}(t)\delta\rho - \frac{\gamma-1}{\gamma}\eta_1)\sqrt{\nu_t} = \tilde{\theta}_{1,t}^*(a)\sqrt{\nu_t} + \tilde{\theta}_{2,t}^*(a)\frac{\sqrt{\nu_t}(h_S S_t + \delta\rho h_\nu)}{O_t} - \eta_1\sqrt{\nu_t} \\
(\tilde{B}(t)\delta\sqrt{1-\rho^2} - \frac{\gamma-1}{\gamma}\tilde{\eta}_{2,t})\sqrt{\nu_t} = \tilde{\theta}_{2,t}^*(a)\frac{\sqrt{\nu_t}h_\nu\delta\sqrt{1-\rho^2}}{O_t} - \tilde{\eta}_{2,t}\sqrt{\nu_t}
\end{cases} \tag{B.16}$$

And thus we find the following optimal strategy in the dual market:

$$\begin{cases} \tilde{\theta}^*_{1,t}(a) = \dfrac{\eta_1}{\gamma} - \dfrac{(\eta_2 + a_t)\rho}{\gamma\sqrt{1-\rho^2}} - \tilde{\theta}^*_{2,t}(a)\dfrac{h_S S_t}{O_t} \\ \tilde{\theta}^*_{2,t}(a) = \left(\tilde{B}(t) + \dfrac{\eta_2 + a_t}{\gamma\delta\sqrt{1-\rho^2}}\right)\dfrac{O_t}{h_\nu} \end{cases} \tag{B.17}$$

B.3 Numerical implementation investment strategy



(a) $\gamma = 2$ (b) $\gamma = 8$ (c) $\gamma = 10$

Figure B.1: Comparison of the optimal strategy in the Heston model between different values of γ for the alternative parameters. We assume $T = 40$ with monthly time steps in the simulations and $W_0 = 1$. Other parameter values are given in Table 3.2.

B.4 Welfare analysis

γ	2	3	4	5	6	7	8	9	10
CE Heston baseline (%)	10.56	6.69	5.30	4.61	4.19	3.92	3.72	3.58	3.47
	(0.19	(0.12)	(0.09)	(0.07)	(0.06)	(0.05)	(0.04)	(0.04)	(0.03)
CE myopic baseline (%)	10.55	6.69	5.30	4.60	4.19	3.92	3.72	3.58	3.47
	(0.18)	(0.11)	(0.08)	(0.06)	(0.05)	(0.04)	(0.04)	(0.03)	(0.03)
CE Heston alternative (%)	806.31	134.31	53.46	30.33	20.62	15.58	12.59	10.65	9.31
	(79.32)	(15.03)	(5.48)	(2.76)	(1.67)	(1.13)	(0.82)	(0.63)	(0.50)
CE myopic alternative (%)	800.22	130.05	51.08	28.85	19.60	14.83	12.01	10.18	8.92
	(68.36)	(12.66)	(4.60)	(2.32)	(1.40)	(0.95)	(0.70)	(0.54)	(0.43)

Table B.1: Rounded certainty equivalents underlying the welfare losses presented in Table 3.3. The baseline parameters can be found in Table 3.1, the alternative parameters in Table 3.2. Standard errors are reported in parentheses. The data is calculated over 10,000 simulations. We assume $T = 40$ with monthly time steps in the simulations and $W_0 = 1$.

True γ \ Measured γ	2	3	4	5	6	7	8	9	10
2	–	8.99 (0.08)	7.40 (0.04)	6.35 (0.03)	5.64 (0.02)	5.15 (0.02)	4.79 (0.01)	4.51 (0.01)	4.30 (0.01)
3	5.20 (0.32)	–	6.29 (0.05)	5.72 (0.03)	5.25 (0.02)	4.89 (0.02)	4.60 (0.01)	4.37 (0.01)	4.19 (0.01)
4	2.53 (0.29)	4.86 (0.20)	–	5.15 (0.04)	4.89 (0.03)	4.64 (0.02)	4.42 (0.02)	4.24 (0.01)	4.09 (0.01)
5	1.46 (0.18)	3.50 (0.23)	4.42 (0.13)	–	4.53 (0.03)	4.39 (0.03)	4.24 (0.02)	4.10 (0.01)	3.98 (0.01)
6	1.01 (0.11)	2.63 (0.21)	3.67 (0.17)	4.10 (0.10)	–	4.15 (0.04)	4.07 (0.02)	3.97 (0.02)	3.88 (0.01)
7	0.77 (0.08)	2.10 (0.17)	3.09 (0.17)	3.64 (0.13)	3.86 (0.08)	–	3.89 (0.03)	3.84 (0.02)	3.77 (0.02)
8	0.64 (0.06)	1.76 (0.13)	2.67 (0.16)	3.25 (0.14)	3.56 (0.10)	3.69 (0.06)	–	3.71 (0.03)	3.67 (0.02)
9	0.55 (0.04)	1.53 (0.11)	2.36 (0.14)	2.93 (0.14)	3.28 (0.11)	3.47 (0.08)	3.56 (0.05)	–	3.57 (0.03)
10	0.49 (0.03)	1.37 (0.09)	2.13 (0.13)	2.68 (0.13)	3.05 (0.11)	3.27 (0.09)	3.39 (0.07)	3.45 (0.05)	–

Table B.2: Rounded certainty equivalents when an agent invests according to the measured γ, whereas the utility is evaluated on the basis of the true γ. The welfare losses in Table 3.4 can be calculated when combining this table with the certainty equivalents for the baseline Heston strategy in Table B.1. The data is calculated over $10,000$ simulations. We assume $T = 40$ with monthly time steps in the simulations and $W_0 = 1$. Other parameter values are given in Table 3.1.

True γ \ Measured γ	2	3	4	5	6	7	8	9	10
2	–	447.29 (14.21)	204.55 (3.49)	109.34 (1.28)	67.39 (0.61)	46.15 (0.34)	34.12 (0.21)	26.68 (0.15)	21.76 (0.11)
3	81.55 (16.88)	–	103.51 (5.39)	71.03 (1.92)	50.05 (0.82)	37.11 (0.43)	28.88 (0.25)	23.38 (0.17)	19.56 (0.12)
4	24.52 (4.78)	52.02 (8.16)	–	45.43 (2.76)	36.66 (1.32)	29.59 (0.67)	24.31 (0.37)	20.43 (0.23)	17.54 (0.15)
5	12.67 (2.19)	28.38 (4.56)	32.01 (4.05)	–	26.99 (1.67)	23.48 (0.96)	20.36 (0.56)	17.77 (0.34)	15.68 (0.22)
6	8.38 (1.29)	19.05 (2.91)	22.22 (2.91)	22.02 (2.32)	–	18.86 (1.12)	17.08 (0.73)	15.43 (0.47)	13.98 (0.31)
7	6.32 (0.87)	14.41 (2.03)	17.06 (2.17)	17.25 (1.88)	16.56 (1.49)	–	14.52 (0.82)	13.47 (0.58)	12.48 (0.40)
8	5.14 (0.64)	11.73 (1.51)	13.99 (1.68)	14.30 (1.53)	13.91 (1.30)	13.29 (1.05)	–	11.89 (0.63)	11.20 (0.47)
9	4.39 (0.50)	10.02 (1.17)	12.01 (1.34)	12.34 (1.27)	12.10 (1.12)	11.66 (0.95)	11.16 (0.78)	–	10.15 (0.50)
10	3.88 (0.40)	8.85 (0.94)	10.63 (1.10)	10.97 (1.06)	10.80 (0.97)	10.47 (0.84)	10.08 (0.72)	9.69 (0.61)	–

Table B.3: Rounded certainty equivalents when an agent invests according to the measured γ, whereas the utility is evaluated on the basis of the true γ. The welfare losses in Table 3.5 can be calculated when combining this table with the certainty equivalents for the baseline Heston strategy in Table B.1. The data is calculated over $10,000$ simulations. We assume $T = 40$ with monthly time steps in the simulations and $W_0 = 1$. Other parameter values are given in Table 3.2.

Appendix C

CP2022

C.1 Financial market

We want to find the dynamics of S_t. Note that X_t^o only presents the dynamics of $\ln S_t$:

$$\begin{aligned} d\ln S_t &= [1\ 0]dX_t^o \\ &= [1\ 0]\begin{bmatrix} r_t+\eta_S \\ \pi_t+\eta_\pi \end{bmatrix}dt - \frac{1}{2}[1\ 0]D\Big(\begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix}\begin{bmatrix} \nu_t & 0_{1\times 4} \\ 0_{4\times 1} & I_4+\nu_t\Gamma_1 \end{bmatrix}\begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix}'\Big)dt \\ &\quad + [1\ 0]\Sigma^{S\Pi}(\Gamma_0+(X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t \\ &= (r_t+\eta_S-\frac{1}{2}\sigma_S'(\Gamma_0+(X_t^s)_1\Gamma)\sigma_S)dt + \sigma_S{}'(\Gamma_0+(X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t \end{aligned} \tag{C.1}$$

An application of Itô's Lemma leads to the dynamics of S_t:

$$\begin{aligned} dS_t &= S_t d\ln S_t + \frac{1}{2}S_t\sigma_S{}'(\Gamma_0+(X_t^s)_1\Gamma)\sigma_S dt \\ &= S_t(r_t+\eta_S)dt + S_t\sigma_S{}'(\Gamma_0+(X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t \end{aligned} \tag{C.2}$$

Then we want to find the dynamics of Π_t. Note that X_t^o only presents the dynamics of $\ln \Pi_t$:

$$\begin{aligned} d\ln \Pi_t &= [0\ 1]dX_t^o \\ &= [0\ 1]\begin{bmatrix} r_t+\eta_S \\ \pi_t+\eta_\pi \end{bmatrix}dt - \frac{1}{2}[0\ 1]D\Big(\begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix}\begin{bmatrix} \nu_t & 0_{1\times 4} \\ 0_{4\times 1} & I_4+\nu_t\Gamma_1 \end{bmatrix}\begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix}'\Big)dt \\ &\quad + [0\ 1]\Sigma^{S\Pi}(\Gamma_0+(X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t \\ &= (\pi_t+\eta_\pi-\frac{1}{2}\sigma_\Pi'(\Gamma_0+(X_t^s)_1\Gamma)\sigma_\Pi)dt + \sigma_\Pi{}'(\Gamma_0+(X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t \end{aligned} \tag{C.3}$$

An application of Itô's Lemma leads to the dynamics of Π_t:

$$\begin{aligned} d\Pi_t &= \Pi_t d\ln \Pi_t + \frac{1}{2}\Pi_t\sigma_\Pi{}'(\Gamma_0+(X_t^s)_1\Gamma)\sigma_\Pi dt \\ &= \Pi_t(\pi_t+\eta_\pi)dt + \Pi_t\sigma_\Pi{}'(\Gamma_0+(X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t \end{aligned} \tag{C.4}$$

We can use the dynamics of Π_t to define the real pricing kernel as $\phi_t^R = \phi_t^N \Pi_t$, whose dynamics are defined as follows:

$$\begin{aligned}
\frac{d\phi_t^R}{\phi_t^R} &= \frac{\phi_t^N d\Pi_t}{\phi_t^N \Pi_t} + \frac{\Pi_t d\phi_t^N}{\phi_t^N \Pi_t} + \frac{d[\phi^N, \Pi]_t}{\phi_t^N \Pi_t} \\
&= \frac{d\Pi_t}{\Pi_t} + \frac{d\phi_t^N}{\phi_t^N} - \sigma_\Pi{}'(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}\Lambda_t dt \\
&= -(r_t - \pi_t - \eta_\pi + \sigma_\Pi{}'(\Lambda_0 + \Lambda_1 X_t^s))dt - (\Lambda_t' - \sigma_\Pi{}'(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}})dZ_t \\
&= -R_t dt - \Lambda_t^{R\prime} dZ_t
\end{aligned} \tag{C.5}$$

Note that we have introduced the real instantaneous interest rate R_t and the real price of risk Λ_t^R above.

Then we want to find the nominal bond price dynamics by deriving an explicit expression of $P_{t+\tau_i}^N(t, X_t^s) = \mathbb{E}\left[\frac{\phi_{t+\tau_i}^N}{\phi_t^N}\middle|\mathcal{F}_t\right]$. Hence, we first need to find an expression for ϕ_t^N. By means of finding the dynamics of $\log \phi_t^N$, integrating these dynamics and taking the exponential we find the following:

$$\phi_t^N = \exp\left\{\int_0^t -\left(r_s + \frac{1}{2}\Lambda_s'\Lambda_s\right)ds - \int_0^t \Lambda_s{}' dZ_s\right\} \tag{C.6}$$

Consequently, note that $\frac{\phi_{t+\tau_i}^N}{\phi_t^N}$ looks as follows:

$$\frac{\phi_{t+\tau_i}^N}{\phi_t^N} = \exp\left\{\int_t^{t+\tau_i} -\left(r_s + \frac{1}{2}\Lambda_s'\Lambda_s\right)ds - \int_t^{t+\tau_i} \Lambda_s{}' dZ_s\right\} \tag{C.7}$$

To show that $P_{t+\tau_i}^N(t, X_t^s)$ is the conditional expectation of an exponentially affine function of the state variables X_t^s, we define the Radon-Nikodym derivative $\xi_t = \frac{dQ}{dP}_{|\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t \Lambda_s'\Lambda_s ds - \int_0^t \Lambda_s' dZ_s\right\}$. With this Radon-Nikodym derivative we can rewrite the conditional expectation of interest to an expectation under the EMM described by ξ_t:

$$\begin{aligned}
P_{t+\tau_i}^N(t, X_t^s) &= \mathbb{E}\left[\frac{\phi_{t+\tau_i}^N}{\phi_t^N}\middle|\mathcal{F}_t\right] = \mathbb{E}^Q\left[\frac{\frac{\phi_{t+\tau_i}^N}{\phi_t^N}}{\xi_t}\middle|X_t^s\right] \\
&= \mathbb{E}^Q\left[\exp\left\{-\int_t^{t+\tau_i} r_s ds\right\}\middle|X_t^s\right]
\end{aligned} \tag{C.8}$$

As $r_t = [0\ 1\ 0]X_t^s$ we see that the conditional expectation in (C.8) is an exponentially affine function of X_t^s. Since the state variables follow a mean-reverting process we can apply the results of Duffie and Kan (1996). Therefore we know that the bond prices are an exponentially affine function of X_t^s: $P_{t+\tau_i}^N(t, X_t^s) = \exp\left(A^N(\tau_i) + B^N(\tau_i)' X_t^s\right)$. As a direct consequence we can apply the Feynman-Kac theorem to our expectation under $\mathbb{Q}$ in (C.8). This leads to the following PDE that describes nominal bond prices:

$$\begin{aligned}0 &= \dot{P}^N_{t+\tau_i}(t, X^s_t) + P_{X^s_t}(t, X^s_t)' K(\mathbb{E}X^s_\infty - X^s_t)\\&+ \frac{1}{2} tr\Big(\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}} P_{X^s_t X^s_t}(\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}})'\Big) - r_t P^N_{t+\tau_i}(t, X^s_t)\\&- P_{X^s_t}(t, X^s_t)'\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}((\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}})^{-1}(\Lambda_0 + \Lambda_1 X^s_t)\end{aligned} \tag{C.9}$$

Note that we express the bond price as a function of the time to maturity τ_i. Since the time to maturity τ_i decreases as t increases, the time derivative of $P^N_{t+\tau_i}(t, X^s_t)$ takes a negative sign when expressed as a function of τ_i. We thus find the following derivatives:

$$\begin{cases}\dot{P}^N_{t+\tau_i}(t, X^s_t) = P^N_{t+\tau_i}(t, X^s_t)(-\dot{A}^N(\tau_i) - \dot{B}^N(\tau_i)' X^s_t)\\ P_{X^s_t}(t, X^s_t) = P^N_{t+\tau_i}(t, X^s_t) B^N(\tau_i)\\ P_{X^s_t X^s_t}(t, X^s_t) = P^N_{t+\tau_i}(t, X^s_t) B^N(\tau_i) B^N(\tau_i)'\end{cases} \tag{C.10}$$

Before continuing note the following:

$$\begin{aligned}&tr\Big((\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}) P_{X^s_t X^s_t}(\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}})'\Big) =\\&tr\Big(P^N_{t+\tau_i}(t, X^s_t)\Sigma^{r\pi}(G_0 + \sum_i (X^s_t)_i G_i)^{\frac{1}{2}} B^N B^{N\prime}(\Sigma^{r\pi}(G_0 + \sum_i (X^s_t)_i G_i)^{\frac{1}{2}})'\Big) =\\&tr\Big(P^N_{t+\tau_i}(t, X^s_t) B^N(\tau_i)'\Sigma^{r\pi}(G_0 + \sum_i (X^s_t)_i G_i)\Sigma^{r\pi\prime} B^N(\tau_i)\Big) =\\&P^N_{t+\tau_i}(t, X^s_t) B^N(\tau_i)'\Sigma^{r\pi}(G_0 + \sum_i (X^s_t)_i G_i)\Sigma^{r\pi\prime} B^N(\tau_i)\end{aligned} \tag{C.11}$$

where we have used that $(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}} = (G_0 + \sum_i (X^s_t)_i G_i)^{\frac{1}{2}}$ and the fact that $P^N_{t+\tau_i}(t, X^s_t) B^N(\tau_i)'\Sigma^{r\pi}(G_0 + \sum_i (X^s_t)_i G_i)\Sigma^{r\pi\prime} B^N(\tau_i) \in \mathbb{R}$. Now we can plug the derivatives from (C.10) in the PDE from (C.9). By separating the terms dependent on X^s_t from the terms not dependent on X^s_t and dividing by $P^N_{t+\tau_i}(t, X^s_t)$, we can find the ODE's that describe $A^N(\tau_i)$ and $B^N(\tau_i)$. To ensure clarity, we first give the ODE's for each entry of $B^N(\tau_i)$ separately. We can do so by using the summation structure of the variance covariance matrix in (C.11) and the fact that $r_t = [0\ 1\ 0] X^s_t$. This leads to the following ODE's:

$$\begin{cases}\dot{A}^N(\tau_i) = B^N(\tau_i)'(K\mathbb{E}X^s_\infty - \Sigma^{r\pi}\Lambda_0) + \frac{1}{2} B^N(\tau_i)'\Sigma^{r\pi} G_0 \Sigma^{r\pi\prime} B^N(\tau_i)\\ \dot{B}^N_1(\tau_i) = \frac{1}{2} B^N(\tau_i)'\Sigma^{r\pi} G_1 \Sigma^{r\pi\prime} B^N(\tau_i) - (B^N(\tau_i)'(K + \Sigma^{r\pi}\Lambda_1))_1\\ \dot{B}^N_2(\tau_i) = -(B^N(\tau_i)'(K + \Sigma^{r\pi}\Lambda_1))_2 - 1\\ \dot{B}^N_3(\tau_i) = -(B^N(\tau_i)'(K + \Sigma^{r\pi}\Lambda_1))_3\end{cases} \tag{C.12}$$

The ODE's in (C.12) clearly show the different structures for each entry of

$B^N(\tau_i)$. We can rewrite (C.12) in matrix notation:

$$\begin{cases} \dot{A}^N(\tau_i) = B^N(\tau_i)'(K\mathbb{E}X^s_\infty - \Sigma^{r\pi}\Lambda_0) + \frac{1}{2}B^N(\tau_i)'\Sigma^{r\pi}G_0\Sigma^{r\pi\prime}B^N(\tau_i) \\ \dot{B}^N(\tau_i) = \frac{1}{2}\tilde{B}^{N\prime}\tilde{\Sigma}^{r\pi}\tilde{G}\tilde{\Sigma}^{r\pi\prime}B^N(\tau_i) - (K' + \Lambda_1'\Sigma^{r\pi\prime})B^N(\tau_i) - [0\ 1\ 0]' \end{cases} \tag{C.13}$$

where we have defined the following matrices:

$$\tilde{B}^N = [B^N(\tau_i)\ 0_{3\times 2}],\ \tilde{\Sigma}^{r\pi} = [\Sigma^{r\pi}\ \Sigma^{r\pi}\ \Sigma^{r\pi}],\ \tilde{G} = \begin{bmatrix} G_1 & 0_{5\times5} & 0_{5\times5} \\ 0_{5\times5} & G_2 & 0_{5\times5} \\ 0_{5\times5} & 0_{5\times5} & G_3 \end{bmatrix} \tag{C.14}$$

As the system in (C.13) involves a matrix Riccati equation, no closed form solution exists. We thus have to resolve to numerical approximations for any implementation. To finalize the introduction of the nominal bond, we want to find its dynamics. An application of Itô's Lemma gives us the following:

$$\begin{aligned} \frac{dP^N_{t+\tau_i}(t, X^s_t)}{P^N_{t+\tau_i}(t, X^s_t)} &= \frac{d\exp\Big(A^N(\tau_i) + B^N(\tau_i)'X^s_t\Big)}{P^N_{t+\tau_i}(t, X^s_t)} \\ &= \frac{\dot{P}dt + P'_{X^s_t}dX^s_t + \frac{1}{2}P_{X^s_tX^s_t}d[X^s_t, X^s_t] + \frac{1}{2}\ddot{P}d[t,t]_t + P_{Xt}d[X,t]_t}{P^N_{t+\tau_i}(t, X^s_t)} \\ &= \Big(- \dot{A}^N(\tau_i) - \dot{B}^N(\tau_i)'X^s_t + B^N(\tau_i)'K(\mathbb{E}X^s_\infty - X^s_t) \\ &\quad + \frac{1}{2}B^N(\tau_i)'\Sigma^{r\pi}(G_0 + \sum_i (X^s_t)_i G_i)\Sigma^{r\pi\prime}B^N(\tau_i)\Big)dt \\ &\quad + B^N(\tau_i)'\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}dZ_t \\ &= \Big(r_t + B^N(\tau_i)'\Sigma^{r\pi}(\Lambda_0 + \Lambda_1 X^s_t)\Big)dt \\ &\quad + B^N(\tau_i)'\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}dZ_t \end{aligned} \tag{C.15}$$

Then we would like to find the dynamics of an inflation-indexed bond. We know that the price of an inflation-indexed bond can be expressed via the real pricing kernel as follows: $P^R_{t+\tau}(t, X_t) = \mathbb{E}\Big[\frac{\phi^R_{t+\tau}}{\phi^R_t}\Big|\mathcal{F}_t\Big]$. First of all, we know that $\frac{\phi^R_{t+\tau}}{\phi^R_t}$ looks as follows:

$$\frac{\phi^R_{t+\tau}}{\phi^R_t} = \exp\Big\{\int_t^{t+\tau} -\Big(R_s + \frac{1}{2}\Lambda^{R\prime}_s\Lambda^R_s\Big)ds - \int_t^{t+\tau}\Lambda^{R\prime}_s dZ_s\Big\} \tag{C.16}$$

To utilize the results by Duffie and Kan (1996), we define the Radon-Nikodym derivative $\xi_t = \frac{dQ}{dP}_{|\mathcal{F}_t} = \exp\Big\{-\frac{1}{2}\int_0^t \Lambda^{R\prime}_s\Lambda^R_s ds - \int_0^t \Lambda^{R\prime}_s dZ_s\Big\}$. With this Radon-Nikodym derivative we can rewrite the conditional expectation which describes the inflation-linked bond price to an expectation under the EMM described by

ξ_t:

$$P^R_{t+\tau}(t, X_t) = \mathbb{E}\Big[\frac{\phi^R_{t+\tau}}{\phi^R_t}\Big|\mathcal{F}_t\Big] = \mathbb{E}^Q\left[\exp\left\{-\int_t^{t+\tau} R_s ds\right\}\Big|X_t\right] \tag{C.17}$$

Since the instantaneous real interest rate R_t is an affine function of X_t, we can apply the results by Duffie and Kan (1996) on affine yield models similarly to how we did for the nominal bonds. Accordingly, we know that the price of the inflation linked bond is an exponentially affine function of X_t: $P^R_{t+\tau}(t, X_t) = \exp\Big(A^R(\tau) + B^R(\tau)'X_t\Big)$. An application of the Feynamn-Kac theorem to the expectation under $\mathbb{Q}$ in (C.17) leads to the following PDE which describes the price of an inflation-linked bond maturing at time τ:

$$\begin{aligned} 0 = {} & \dot{P}^R_{t+\tau}(t, X_t) + P_{X_t}(t, X_t)'(M + LX_t) \\ & + \frac{1}{2}tr\Big(\Sigma(\Gamma_0 + (X_t)_1\Gamma)^{\frac{1}{2}} P_{X_tX_t}(\Sigma(\Gamma_0 + (X_t)_1\Gamma)^{\frac{1}{2}})'\Big) - R_t P^R_{t+\tau}(t, X_t) \\ & - P_{X_t}(t, X_t)'\Sigma(\Gamma_0 + (X_t)_1\Gamma)^{\frac{1}{2}}\Big(((\Gamma_0 + (X_t)_1\Gamma)^{\frac{1}{2}})^{-1}(\Lambda_0 + \bar{\Lambda}_1 X_t) \\ & - (\Gamma_0 + (X_t)_1\Gamma)^{\frac{1}{2}\prime}\sigma_\Pi\Big) \end{aligned} \tag{C.18}$$

The goal is then to separate the terms multiplied with X_t from the terms not multiplied with X_t and to divide by $P^R_{t+\tau}(t, X_t)$. We can rewrite the term involving the trace in (C.18) similarly to how we have rewritten (C.11). This enables us to use the structure in which the stochastic volatility matrix is presented as a summation of the variables in X_t. Therefore, the only term in (C.18) that remains to be specified as function of X_t is the instantaneous real interest rate R_t. With this in mind, note that $R_t = r_t - \pi_t - \eta_\pi + \sigma_\Pi'(\Lambda_0 + \Lambda_1 X^s_t)$. Hence, we use the following to rewrite R_t:

$$\begin{cases} r_t = [0\ 1\ 0\ 0\ 0]X_t \\ \pi_t = [0\ 0\ 1\ 0\ 0]X_t \\ \Lambda_1 X^s_t = \bar{\Lambda}_1 X_t \end{cases} \tag{C.19}$$

By using (C.19) all elements of the PDE in (C.18) are expressed in terms of X_t. To ease the interpretation of the ODE's that follow from (C.18), we first present them for each entry of $B^R(\tau)$ separately:

$$\begin{cases} \dot{A}^R(\tau) = B^R(\tau)'\Big(M - \Sigma(\Lambda_0 - G_0\sigma_\Pi)\Big) + \frac{1}{2}B^R(\tau)'\Sigma G_0 \Sigma' B^R(\tau) + \eta_\pi \\ \qquad\qquad - \sigma_\Pi' \Lambda_0 \\ \dot{B}_1^R(\tau) = \frac{1}{2}B^R(\tau)'\Sigma G_1 \Sigma' B^R(\tau) + (B^R(\tau)'L)_1 - (\sigma_\Pi'\bar{\Lambda}_1)_1 \\ \qquad\qquad - (B^R(\tau)'\Sigma\bar{\Lambda}_1)_1 + B^R(\tau)'\Sigma G_1 \sigma_\Pi \\ \dot{B}_2^R(\tau) = (B^R(\tau)'L)_2 - (\sigma_\Pi'\bar{\Lambda}_1)_2 - (B^R(\tau)'\Sigma\bar{\Lambda}_1)_2 - 1 \\ \dot{B}_3^R(\tau) = (B^R(\tau)'L)_3 - (\sigma_\Pi'\bar{\Lambda}_1)_3 - (B^R(\tau)'\Sigma\bar{\Lambda}_1)_3 + 1 \\ \dot{B}_4^R(\tau) = (B^R(\tau)'L)_4 - (\sigma_\Pi'\bar{\Lambda}_1)_4 - (B^R(\tau)'\Sigma\bar{\Lambda}_1)_4 \\ \dot{B}_5^R(\tau) = (B^R(\tau)'L)_5 - (\sigma_\Pi'\bar{\Lambda}_1)_5 - (B^R(\tau)'\Sigma\bar{\Lambda}_1)_5 \end{cases} \tag{C.20}$$

Note that the last two columns of L and $\bar{\Lambda}_1$ are zero. Consequently we see from (C.20) that $\dot{B}_4^R(\tau) = \dot{B}_5^R(\tau) = 0$. Combining this with the boundary condition $B^R(0) = 0_{5\times1}$ leads to the conclusion that $B_4^R(\tau) = B_5^R(\tau) = 0$. Hence, we know that we can remove the dependency on X_t^o from the bond prices. Therefore, we rewrite the bond prices as follows: $P_{t+\tau}^R(t, X_t) = P_{t+\tau}^R(t, X_t^s) = \exp\Big(A^R(\tau) + B^R(\tau)'X_t^s\Big)$, with $A^R(\tau) \in \mathbb{R}$ and $B^R(\tau) \in \mathbb{R}^{3\times1}$. Consequently, we rewrite the ODE's in (C.20) in matrix notation:

$$\begin{cases} \dot{A}^R(\tau) = B^R(\tau)'\Big(K\mathbb{E}X_\infty^s - \Sigma^{r\pi}(\Lambda_0 - G_0\sigma_\Pi)\Big) + \frac{1}{2}B^R(\tau)'\Sigma^{r\pi}G_0\Sigma^{r\pi\prime}B^R(\tau) \\ \qquad\qquad + \eta_\pi - \sigma_\Pi'\Lambda_0 \\ \dot{B}^R(\tau) = \frac{1}{2}\tilde{B}^{R\prime}\tilde{\Sigma}^{r\pi}\tilde{G}\tilde{\Sigma}^{r\pi}\tilde{\Sigma}^{r\pi\prime}B^R - (K' + \Lambda_1'\Sigma^{r\pi\prime} - \tilde{\sigma}_\Pi'\tilde{G}'\tilde{\Sigma}^{r\pi\prime})B^R(\tau) \\ \qquad\qquad - \Lambda_1'\sigma_\Pi - [0\ 1\ \ -1]' \end{cases} \tag{C.21}$$

where we have used the definitions of $\tilde{\Sigma}^{r\pi}$ and $\tilde{G}$ from (C.14) and we have defined the following matrices:

$$\tilde{B}^R = [B^R(\tau) \quad 0_{3\times2}], \quad \tilde{\sigma}_\Pi = \begin{bmatrix} \sigma_\Pi & 0_{5\times2} \\ \sigma_\Pi & 0_{5\times2} \\ \sigma_\Pi & 0_{5\times2} \end{bmatrix} \tag{C.22}$$

Similar as for the nominal bond prices the system in (C.21) involves a matrix Riccati equation and thus no closed form solution exists. Finally, we are interested in the dynamics of $P^R_{t+\tau}(t, X^s_t) = \exp\Big(A^R(\tau) + B^R(\tau)' X^s_t\Big)$:

$$
\begin{aligned}
\frac{dP^R_{t+\tau}(t,X^s_t)}{P^R_{t+\tau}(t,X^s_t)} &= \frac{d\exp\Big(A^R(\tau) + B^R(\tau)'X^s_t\Big)}{P^R_{t+\tau}(t,X^s_t)} \\
&= \frac{\dot{P}dt + P'_{X^s_t}dX^s_t + \frac{1}{2}P_{X^s_tX^s_t}d[X^s_t, X^s_t] + \frac{1}{2}\ddot{P}d[t,t]_t + P_{Xt}d[X,t]_t}{P^R_{t+\tau}(t,X^s_t)} \\
&= \Big(-\dot{A}^R(\tau) - \dot{B}^R(\tau)'X^s_t + B^R(\tau)'K(\mathbb{E}X^s_\infty - X^s_t) \\
&\quad + \frac{1}{2}B^R(\tau)'\Sigma^{r\pi}(G_0 + \sum_i (X^s_t)_i G_i)\Sigma^{r\pi\prime}B^R(\tau)\Big)dt \\
&\quad + B^R(\tau)'\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}dZ_t \\
&= \Big(R_t + B^R(\tau)'\Sigma^{r\pi}(\Lambda_0 + \Lambda_1 X^s_t - (\Gamma_0 + (X^s_t)_1\Gamma)\sigma_\Pi)\Big)dt \\
&\quad + B^R(\tau)'\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}dZ_t
\end{aligned}
\tag{C.23}
$$

where we have used (C.18) in combination with the fact that $B^R_4(\tau) = B^R_5(\tau) = 0$. Lastly, we would like to find the price dynamics of the real bond expressed in nominal terms: $\hat{P}^R_{t+\tau}(t, X^s_t) = P^R_{t+\tau}(t, X^s_t)\Pi_t$. An application of Itô's Lemma yields:

$$
\begin{aligned}
\frac{d\hat{P}^R_{t+\tau}(t,X^s_t)}{\hat{P}^R_{t+\tau}(t,X^s_t)} &= \frac{P^R_{t+\tau}(t,X^s_t)d\Pi_t}{P^R_{t+\tau}(t,X^s_t)\Pi_t} + \frac{\Pi_t dP^R_{t+\tau}(t,X^s_t)}{P^R_{t+\tau}(t,X^s_t)\Pi_t} + \frac{d[P^R_{t+\tau}(t,X^s_t),\Pi_t]}{P^R_{t+\tau}(t,X^s_t)\Pi_t} \\
&= \frac{d\Pi_t}{\Pi_t} + \frac{dP^R_{t+\tau}(t,X^s_t)}{P^R_{t+\tau}(t,X^s_t)} + B^R(\tau)'\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)\sigma_\Pi dt \\
&= \Big(R_t + B^R(\tau)'\Sigma^{r\pi}(\Lambda_0 + \Lambda_1X^s_t - (\Gamma_0 + (X^s_t)_1\Gamma)\sigma_\Pi) + \mu_\Pi \\
&\quad + B^R(\tau)'\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)\sigma_\Pi\Big)dt \\
&\quad + \Big(B^R(\tau)'\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}} + \sigma'_\Pi(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}\Big)dZ_t \\
&= \Big(r_t + (B^R(\tau)'\Sigma^{r\pi} + \sigma'_\Pi)(\Lambda_0 + \Lambda_1 X^s_t)\Big)dt \\
&\quad + \Big((B^R(\tau)'\Sigma^{r\pi} + \sigma'_\Pi)(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}\Big)dZ_t
\end{aligned}
\tag{C.24}
$$

The asset mix is concluded with an asset that trades the stochastic volatility risk. We define the price of this asset as $O_t = h(t, S_t, \nu_t)$. Before applying Itô's Lemma to O_t, we first present the dynamics of ν_t:

$$
\begin{aligned}
d\nu_t &= [1\ 0\ 0]dX^s_t \\
&= [1\ 0\ 0]K(\mathbb{E}X^s_\infty - X^s_t)dt + [1\ 0\ 0]\Sigma^{r\pi}(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}dZ_t \\
&= K_{\nu\nu}(\mathbb{E}\nu_\infty - \nu_t)dt + \sigma'_\nu(\Gamma_0 + (X^s_t)_1\Gamma)^{\frac{1}{2}}dZ_t
\end{aligned}
\tag{C.25}
$$

where we thus have that $\sigma_\nu = [\omega\ 0\ 0\ 0\ 0]'$. Then we can apply Itô's Lemma to

$h(t, S_t, \nu_t)$:

$$
\begin{aligned}
dO_t &= \dot{h}dt + h_S dS_t + h_\nu d\nu_t + \frac{1}{2}h_{SS}d[S,S]_t + \frac{1}{2}h_{\nu\nu}d[\nu,\nu]_t + h_{S\nu}d[S,\nu]_t \\
&= \dot{h}dt + h_S(S_t(r_t + \eta_S)dt + S_t\sigma_S'(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}) \\
&\quad + h_\nu(K_{\nu\nu}(\mathbb{E}\nu_\infty - \nu_t)dt + \sigma_\nu'(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t) \\
&\quad + \frac{1}{2}h_{SS}\sigma_S'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_S dt + \frac{1}{2}h_{\nu\nu}\sigma_\nu'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\nu dt \\
&\quad + h_{S\nu}S_t\sigma_S'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\nu dt \\
&= \Big(\dot{h} + h_S S_t(r_t + \eta_S) + h_\nu K_{\nu\nu}(\mathbb{E}\nu_\infty - \nu_t) + \frac{1}{2}h_{SS}\sigma_S'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_S \\
&\quad + \frac{1}{2}h_{\nu\nu}\sigma_\nu'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\nu + h_{S\nu}S_t\sigma_S'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\nu\Big)dt \\
&\quad + (h_S S_t\sigma_S' + h_\nu\sigma_\nu')(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t
\end{aligned}
\tag{C.26}
$$

We can simplify (C.26) by using the martingale property of $\phi_t^N O_t$. Therefore, we first find the dynamics of $\phi_t^N O_t$:

$$
\begin{aligned}
d\phi_t^N O_t &= \phi_t^N dO_t + O_t d\phi_t^N + d[\phi_t^N, O_t] \\
&= \phi_t^N\bigg(\Big(\dot{h} + h_S S_t(r_t + \eta_S) + h_\nu K_{\nu\nu}(\mathbb{E}\nu_\infty - \nu_t) + \frac{1}{2}h_{SS}\sigma_S'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_S \\
&\quad + \frac{1}{2}h_{\nu\nu}\sigma_\nu'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\nu + h_{S\nu}S_t\sigma_S'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\nu\Big)dt \\
&\quad + (h_S S_t\sigma_S' + h_\nu\sigma_\nu')(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t\bigg) + O_t\Big(-\phi_t^N r_t dt - \phi_t^N\Lambda_t' dZ_t\Big) \\
&\quad - \phi_t^N(h_S S_t\sigma_S' + h_\nu\sigma_\nu')(\Lambda_0 + \Lambda_1 X_t^s)dt \\
&= \phi_t^N\Big(\dot{h} + h_S S_t(r_t + \eta_S) + h_\nu K_{\nu\nu}(\mathbb{E}\nu_\infty - \nu_t) + \frac{1}{2}h_{SS}\sigma_S'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_S \\
&\quad + \frac{1}{2}h_{\nu\nu}\sigma_\nu'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\nu + h_{S\nu}S_t\sigma_S'(\Gamma_0 + (X_t^s)_1\Gamma)\sigma_\nu - O_t r_t \\
&\quad - (h_S S_t\sigma_S' + h_\nu\sigma_\nu')(\Lambda_0 + \Lambda_1 X_t^s)\Big)dt \\
&\quad + \phi_t^N\Big((h_S S_t\sigma_S' + h_\nu\sigma_\nu')(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}} - O_t\Lambda_t'\Big)dZ_t
\end{aligned}
\tag{C.27}
$$

By FTAP we know that $\phi_t^N O_t$ is a martingale. Therefore its drift term should equate to zero. Hence, we can simplify (C.26) to the following:

$$
\begin{aligned}
dO_t = &\Big(r_t O_t + (h_S S_t\sigma_S' + h_\nu\sigma_\nu')(\Lambda_0 + \Lambda_1 X_t^s)\Big)dt \\
&+ (h_S S_t\sigma_S' + h_\nu\sigma_\nu')(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t
\end{aligned}
\tag{C.28}
$$

C.2 Portfolio optimization problem

We are interested in finding an expression for $\tilde{P}(t, X_t) = \mathbb{E}\left[(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}|\mathcal{F}_t\right]$. This expectation is similar to the conditional expectation we have solved in Appendix A.2 for the KNW model. Hence, we know that we can write $(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}$ as follows:

$$(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}} = \exp\left\{\int_t^T -(\frac{\gamma-1}{\gamma}\tilde{R}_s + \frac{1}{2}\frac{\gamma-1}{\gamma}\tilde{\Lambda}_s^{R\prime}\tilde{\Lambda}_s^R)ds - \int_t^T \frac{\gamma-1}{\gamma}\tilde{\Lambda}_s^{R\prime}dZ_s\right\} \tag{C.29}$$

Hence, if we define the EMM $\mathbb{Q}$ described by the Radon-Nikodym derivative $\xi_t = \frac{dQ}{dP}_{|\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t \lambda_s'\lambda_s ds - \int_0^t \lambda_s' dZ_s\right\}$ where $\lambda_t = \frac{\gamma-1}{\gamma}\tilde{\Lambda}_t^R$, we can rewrite our conditional expectation of interest to an expectation under the EMM described by ξ_t:

$$\begin{aligned}\tilde{P}(t, X_t) &= \mathbb{E}\left[(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}\middle|\mathcal{F}_t\right] = \mathbb{E}^Q\left[\frac{(\frac{\tilde{\phi}_T^R}{\tilde{\phi}_t^R})^{\frac{\gamma-1}{\gamma}}}{\xi_t}\middle|X_t\right] \\ &= \mathbb{E}^Q\left[\exp\left\{-\int_t^T\left(\frac{\gamma-1}{\gamma}\tilde{R}_s + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_s^{R\prime}\tilde{\Lambda}_s^R\right)ds\right\}\middle|X_t\right]\end{aligned} \tag{C.30}$$

C.3 Possible alternative investment strategies

We are interested in finding the function that solves the modified PDE given below:

$$\begin{aligned}0 = \dot{\tilde{P}} &+ \tilde{P}_{X_t}'(M + LX_t) + \frac{1}{2}\operatorname{tr}\left(\Sigma(\tilde{G}^\infty)^{\frac{1}{2}}\tilde{P}_{X_tX_t}(\Sigma(\tilde{G}^\infty)^{\frac{1}{2}})'\right) \\ &- \left(\frac{\gamma-1}{\gamma}R_t + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\tilde{\Lambda}_t^{R\prime}\tilde{\Lambda}_t^R\right)\tilde{P} - \frac{\gamma-1}{\gamma}\tilde{P}_{X_t}'\Sigma(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}\Lambda_t^R\end{aligned} \tag{C.31}$$

We have assumed the form $\tilde{P}(t, X_t) = \exp\left(\tilde{A}(t) + \tilde{B}(t)'X_t + X_t'\tilde{C}(t)X_t\right)$. If we plug the corresponding derivatives in the PDE from (C.31), we find the following:

$$\begin{aligned}
0 =& \dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)'X_t + X_t'\dot{\tilde{C}}(t)X_t + (\tilde{B}(t) + 2\tilde{C}(t)X_t)'(M + LX_t) \\
&+ \frac{1}{2}\tilde{B}(t)'\Sigma\tilde{G}^\infty\Sigma'\tilde{B}(t) + 2\tilde{B}(t)'\Sigma\tilde{G}^\infty\Sigma'\tilde{C}(t)X_t + 2X_t'\tilde{C}(t)'\Sigma\tilde{G}^\infty\Sigma'\tilde{C}(t)X_t \\
&+ \mathrm{tr}\left(\Sigma(\tilde{G}^\infty)^{\frac{1}{2}}\tilde{C}(t)(\Sigma(\tilde{G}^\infty)^{\frac{1}{2}})'\right) - \left(\frac{\gamma-1}{\gamma}\Big([0\ 1\ 0\ 0\ 0]X_t - [0\ 0\ 1\ 0\ 0]X_t - \eta_\pi\right. \\
&+ \sigma_\Pi'\Lambda_0 + \sigma_\Pi'\bar{\Lambda}_1\Big) + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\Big(\Lambda_0'(\tilde{G}^\infty)^{-1}\Lambda_0 + 2\Lambda_0'(\tilde{G}^\infty)^{-1}\bar{\Lambda}_1X_t \\
&\left. + X_t'\bar{\Lambda}_1'(\tilde{G}^\infty)^{-1}\bar{\Lambda}_1X_t - 2\sigma_\Pi'(\Lambda_0 + \bar{\Lambda}_1X_t) + \sigma_\Pi'\tilde{G}^\infty\sigma_\Pi\Big)\right) \\
&- \frac{\gamma-1}{\gamma}(\tilde{B}(t) + 2\tilde{C}(t)X_t)'\Sigma(\Lambda_0 + \bar{\Lambda}_1X_t - (G_0 + G_1X_{1,t})\sigma_\Pi)
\end{aligned} \tag{C.32}$$

We can separate the PDE above in three parts: a term independent of X_t, a term affine in X_t and a term that is affine quadratic in X_t. Since all three parts should equate to zero, we can simplify the PDE above to a system of ODE's describing $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$:

$$\begin{aligned}
\dot{\tilde{A}}(t) =& -\tilde{B}(t)M - \frac{1}{2}\tilde{B}(t)'\Sigma\tilde{G}^\infty\Sigma'\tilde{B}(t) - \mathrm{tr}\left(\Sigma(\tilde{G}^\infty)^{\frac{1}{2}}\tilde{C}(t)(\Sigma(\tilde{G}^\infty)^{\frac{1}{2}})'\right) \\
&- \frac{\gamma-1}{\gamma}\Big(\eta_\pi - \sigma_\Pi'\Lambda_0 - \tilde{B}(t)'\Sigma\Lambda_0 - \tilde{B}(t)'\Sigma G_0\sigma_\Pi\Big) \\
&+ \frac{1}{2}\frac{\gamma-1}{\gamma^2}\Big(\Lambda_0'\tilde{G}^\infty)^{-1}\Lambda_0 - 2\sigma_\Pi'\Lambda_0 + \sigma_\Pi'\tilde{G}^\infty\sigma_\Pi\Big) \\
\dot{\tilde{B}}(t) =& -\Big(L' + 2\tilde{C}(t)'\Sigma\tilde{G}^\infty\Sigma' - \frac{\gamma-1}{\gamma}\bar{\Lambda}_1'\Sigma' + \frac{\gamma-1}{\gamma}\hat{\sigma}_\Pi\hat{G}'\hat{\Sigma}'\Big)\tilde{B}(t) \\
&+ \frac{\gamma-1}{\gamma}\Big([0\ 1\ 0\ 0\ 0]' - [0\ 0\ 1\ 0\ 0]'\bar{\Lambda}_1'\sigma_\Pi + 2\tilde{C}(t)'\Sigma\Lambda_0 - 2\tilde{C}(t)'\Sigma G_0\sigma_\Pi\Big) \\
&+ \frac{\gamma-1}{\gamma^2}\Big(\bar{\Lambda}_1'(\tilde{G}^\infty)^{-1}\Lambda_0\Big) \\
\dot{\tilde{C}}(t) =& -2\tilde{C}(t)'\Sigma\tilde{G}^\infty\Sigma'\tilde{C}(t) - 2\Big(L' - \frac{\gamma-1}{\gamma}\bar{\Lambda}_1'\Sigma' + \frac{\gamma-1}{\gamma}\hat{\sigma}'_{Pi}\hat{G}'\hat{\Sigma}'\Big)\tilde{C}(t) \\
&+ \frac{1}{2}\frac{\gamma-1}{\gamma^2}\bar{\Lambda}_1'(\tilde{G}^\infty)^{-1}\bar{\Lambda}_1
\end{aligned} \tag{C.33}$$

where we have defined the following matrices:

$$\hat{\sigma}_\Pi = \begin{bmatrix} \sigma_\Pi & 0_{5\times4} \\ \sigma_\Pi & 0_{5\times4} \\ \sigma_\Pi & 0_{5\times4} \\ \sigma_\Pi & 0_{5\times4} \\ \sigma_\Pi & 0_{5\times4} \end{bmatrix},\ \hat{\Sigma} = [\Sigma\ \Sigma\ \Sigma\ \Sigma\ \Sigma],\ \hat{G} = \begin{bmatrix} G_1 & 0_{5\times20} \\ 0_{20\times5} & 0_{20\times20} \end{bmatrix} \tag{C.34}$$

We know that $\mathbb{E}\left[(\frac{\phi_T^R}{\phi_T^R})^{\frac{\gamma-1}{\gamma}}|\mathcal{F}_t\right] = 1$ and thus we assume the terminal conditions to be $\tilde{A}(T) = 0$, $\tilde{B}(T) = 0_{5\times1}$ and $\tilde{C}(T) = 0_{5\times5}$. Note that no closed form solution is available for the system since it involves a matrix Riccati equation. Furthermore we state that the last two rows of $\tilde{B}(t)$ and the last two rows and columns of $\tilde{C}(t)$ will be zero by the construction of the system in (C.33). Hence, the conditional expectation will in practice only depend on X_t^s.

Then we are interest in finding the dynamics of our estimated optimal wealth $\tilde{W}_t^* \phi_t^N = \frac{W_0}{g_T}(\phi_t^R)^{1-\frac{1}{\gamma}} \tilde{P}(t, X_t)$. To find these dynamics we first derive the dynamics of $\tilde{P}(t, X_t) = \exp\Big(\tilde{A}(t) + \tilde{B}(t)' X_t + X_t' \tilde{C}(t) X_t\Big)$:

$$
\begin{aligned}
d\tilde{P}(t, X_t) &= \dot{\tilde{P}} dt + \tilde{P}_X dX_t + \frac{1}{2} \tilde{P}_{XX} d[X, X]_t \\
&= \tilde{P}(t, X_t)\{\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)' X_t + X_t' \dot{\tilde{C}} X_t + (\tilde{B}(t) + 2\tilde{C}(t) X_t)'(M + L X_t) \\
&\quad + \frac{1}{2} \operatorname{tr}\Big(\Sigma(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}} \tilde{P}_{X_t X_t} (\Sigma(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}})'\Big)\} dt \\
&\quad + \tilde{P}(t, X_t)(\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma(G_0 + \sum_i X_t^i G_i)^{\frac{1}{2}} dZ_t
\end{aligned}
\tag{C.35}
$$

If we would have the true representation of $P(t, X_t)$ we know that $W_t \phi_t^N$ is be a martingale. Since we use an estimation for $P(t, X_t)$ we assume that $\tilde{W}_t^* \phi_t^N$ is approximately a martingale. In other words, we take the modifications we have done to the PDE into account when determining the dynamics of $\tilde{W}_t^* \phi_t^N$. Hence, we modify the dynamics of $\tilde{W}_t^* \phi_t^N$ similarly to how we modified the PDE in (C.31). Therefore we find the following dynamics of $\tilde{W}_t^* \tilde{\phi}_t^N$, with slight abuse of its martingale property:

$$
\begin{aligned}
d\tilde{W}_t^* \phi_t^N &= d\frac{W_0}{g_T}(\phi_t^R)^{\frac{\gamma-1}{\gamma}} \tilde{P}(t, X_t) \\
&= \frac{W_0}{g_T}\Big((\phi_t^R)^{\frac{\gamma-1}{\gamma}} d\tilde{P}(t, X_t) + \tilde{P}(t, X_t) d(\phi_t^R)^{\frac{\gamma-1}{\gamma}} + d\Big[(\phi_t^R)^{\frac{\gamma-1}{\gamma}}, \tilde{P}(t, X_t)\Big]\Big) \\
&= \frac{W_0}{g_T}(\phi_t^R)^{\frac{\gamma-1}{\gamma}} \tilde{P}(t, X_t)\Big(\dot{\tilde{A}}(t) + \dot{\tilde{B}}(t)' X_t + X_t' \dot{\tilde{C}}(t) X_t \\
&\quad + (\tilde{B}(t) + 2\tilde{C}(t) X_t)'(M + L X_t) \\
&\quad + \frac{1}{2} \operatorname{tr}\Big(\Sigma(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}} \tilde{P}_{X_t X_t} (\Sigma(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}})'\Big) - (\frac{\gamma - 1}{\gamma} R_t \\
&\quad + \frac{1}{2} \frac{\gamma - 1}{\gamma^2} \tilde{\Lambda}_t^{R\prime} \tilde{\Lambda}_t^R) - \frac{\gamma - 1}{\gamma} (\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}} \Lambda_t^R\Big) dt \\
&\quad + \Big((\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}} - \frac{\gamma - 1}{\gamma} \Lambda_t^{R\prime}\Big) dZ_t \\
&= \tilde{W}_t^* \tilde{\phi}_t^N \Big((\tilde{B}(t) + 2\tilde{C}(t) X_t)' \Sigma(\Gamma_0 + (X_t^s)_1 \Gamma)^{\frac{1}{2}} - \frac{\gamma - 1}{\gamma} \Lambda_t^{R\prime}\Big) dZ_t
\end{aligned}
\tag{C.36}
$$

We thus assume a modification of the trace operator and the inner product of the price of risk in the drift term in (C.36). Consequently, we can cancel the drift term so that we find a martingale.

Then we find a second way to describe $\tilde{W}_t^* \phi_t^N$, based on the dynamic budget constraint. Note that the budget constraint reads as follows:

$$d\tilde{W}_t^* = \tilde{W}_t^*(r_t + \tilde{\theta}_t^{*\prime}\Sigma_t(\Lambda_0 + \Lambda_1 X_t^s))dt + \tilde{W}_t^*\tilde{\theta}_t^{*\prime}\Sigma_t(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t \quad \text{(C.37)}$$

Hence, we find the following alternative dynamics of $\tilde{W}_t^*\phi_t^N$:

$$\begin{aligned} d\tilde{W}_t^*\phi_t^N &= \tilde{W}_t^* d\phi_t^N + \phi_t^N d\tilde{W}_t^* + d[\tilde{W}_t^*, \phi_t^N] \\ &= \tilde{W}_t^*\phi_t^N(-r_t dt - \Lambda_t' dZ_t) + \tilde{W}_t^*\phi_t^N \Big((r_t + \tilde{\theta}_t^{*\prime}\Sigma_t(\Lambda_0 + \Lambda_1 X_t^s))dt \\ &\quad + \tilde{\theta}_t^{*\prime}\Sigma_t(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}}dZ_t\Big) - \tilde{W}_t^*\phi_t^N\tilde{\theta}_t^{*\prime}\Sigma_t(\Lambda_0 + \Lambda_1 X_t^s)dt \\ &= \tilde{W}_t^*\phi_t^N(\tilde{\theta}_t^{*\prime}\Sigma_t(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}} - \Lambda_t')dZ_t \end{aligned} \quad \text{(C.38)}$$

By equating the volatility terms in (C.36) and (B.14) we arrive at the following equation that defines the estimated strategy in the complete market:

$$(\tilde{B}(t) + 2\tilde{C}(t)X_t)'\Sigma(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}} - \frac{\gamma - 1}{\gamma}\Lambda_t^{R\prime} = \tilde{\theta}_t^{*\prime}\Sigma_t(\Gamma_0 + (X_t^s)_1\Gamma)^{\frac{1}{2}} - \Lambda_t' \quad \text{(C.39)}$$

We therefore find the following estimated strategy:

$$\begin{aligned} \tilde{\theta}_t^* = &\frac{1}{\gamma}\left(\Sigma_t^{-1}\right)^\top (\Gamma_0 + (X_t^s)_1\Gamma)^{-1}(\Lambda_0 + \Lambda_1 X_t^s) + \frac{\gamma - 1}{\gamma}\left(\Sigma_t^{-1}\right)^\top \sigma_\Pi \\ &+ \left(\Sigma_t^{-1}\right)^\top \Sigma^\top \Big(\tilde{B}(t) + 2\tilde{C}(t)X_t\Big) \end{aligned} \quad \text{(C.40)}$$